

# Eikonal propagators and high-energy parton-parton scattering in gauge theories

Enrico Meggiolaro

Dipartimento di Fisica,  
Università di Pisa,  
Via Buonarroti 2,  
I-56127 Pisa, Italy.

## Abstract

In this paper we consider “soft” high-energy parton-parton scattering processes in gauge theories, i.e., elastic scattering processes involving partons at very high squared energies  $s$  in the center of mass and small squared transferred momentum  $t$  ( $s \rightarrow \infty$ ,  $t \ll s$ , typically  $|t| \leq 1 \text{ GeV}^2$ ). By a direct resummation of perturbation theory in the limit we are considering, we derive expressions for the truncated-connected quark (antiquark) propagator in an external gluon field, as well as for the residue at the pole of the full unrenormalized propagator, both for scalar and fermion gauge theories. These are the basic ingredients to derive high-energy parton-parton scattering amplitudes, using the LSZ reduction formulae and a functional integral approach. The above procedure is also extended to include the case in which at least one of the partons is a gluon. The meaning and the validity of the results are discussed.

## 1. Introduction

Since the late 1950s a lot of models have been proposed to describe the physics of hadron–hadron elastic scattering at high energies. Some of these are “pre-QCD” models (see, e.g., Refs. [1] and [2]), while others are “QCD-inspired” models (see, e.g., Ref. [3]). With the advent of Quantum Chromo–Dynamics (QCD), which is now believed to be the correct theory of hadrons and their interactions, many theoretical physicists started to study the high–energy behaviour of gauge theories directly from their first principles. In particular, a lot of work has been done within the framework of perturbation theory in order to find systematic procedures for extracting the high–energy behaviour of each amplitude and for summing these contributions using a leading–log or eikonal approximation scheme [4, 5]. Even if partially successful, the results obtained using these procedures are not completely satisfactory and are not able to explain the most relevant phenomena.

In particular, for “soft” high–energy scattering processes, i.e., elastic scattering processes at very high squared energies  $s$  in the center of mass and small squared transferred momentum  $t$  ( $s \rightarrow \infty$ ,  $t \ll s$ , typically  $|t| \leq 1 \text{ GeV}^2$ ), QCD perturbation theory cannot be safely applied, since  $t$  is too small, and one has therefore to appeal to nonperturbative QCD. P.V. Landshoff and O. Nachtmann were maybe the first who argued [6] that the theoretical description of measurable quantities of “soft” high–energy reactions (like the total cross sections, for example) should involve in an essential way nonperturbative QCD. Later on, Nachtmann developed a nonperturbative analysis, based on QCD, of these “soft” high–energy scattering processes [7, 8]: he derived formal expressions for the quark–quark (and also quark–antiquark and antiquark–antiquark) scattering amplitudes in the above–mentioned limit, by using a functional integral approach and an eikonal approximation to the solution of the Dirac equation in the presence of an external non–Abelian gauge field.

In a previous paper [9] we proposed an alternative approach to high–energy quark–quark scattering based on a first–quantized path–integral description of quantum–field theory developed by Fradkin in the early 1960s [10]. In this approach one obtains convenient expressions for the truncated–connected scalar propagators in an external (gravitational, electromagnetic, etc.) field, and the eikonal approximation can be easily recovered in the relevant limit. Knowing the truncated–connected propagators, one can then extract, in the manner of Lehmann, Symanzik and Zimmermann (LSZ) [11], the scattering matrix elements in the framework of a functional integral approach. (We remind the reader that this method was originally adopted in Ref. [12] in order to study Planckian–

energy gravitational scattering. In Ref. [9] we have translated the procedure and some results of Ref. [12] to the case of Quantum Chromo-Dynamics, i.e., the case of quarks coupled to an external non-Abelian gauge field in a flat space-time.)

In this paper we shall address the same problem in an even more immediate way: by a direct resummation of perturbation theory in the high-energy limit that we are considering, we shall derive expressions for the truncated-connected quark (antiquark) propagator in an external gluon field, as well as for the residue at the pole of the full unrenormalized propagator. (This last is also a basic ingredient to derive parton-parton scattering amplitudes, since it appears in the LSZ reduction formulae [11].)

The paper is organized as follows. In Sect. 2 we begin, for simplicity, with the case of scalar QCD, i.e., the case of a spin-0 quark coupled to a non-Abelian gauge field. In Sect. 3 we extend the results to the case (more interesting from the physical point of view) of “real” fermion QCD: that is a spin-1/2 quark coupled to a non-Abelian gauge field. In Sect. 4 the above procedure is also extended to include the case in which at least one of the partons is a gluon: the meaning and the validity of the results so obtained are discussed. The truncated-connected propagators in an external gluon field are the basic ingredients to derive high-energy parton-parton scattering amplitudes, using the LSZ reduction formulae and a functional integral approach. This was done in Ref. [9] and it will be quickly reviewed in Sect. 5 for the convenience of the reader. The results obtained in Sect. 5 are in agreement with those of Refs. [7, 8, 9], where they were derived with different methods. In Sect. 6 we give a summary of the main results, a detailed discussion of the approximations involved and the conclusions.

## 2. The scattering of scalar quarks and antiquarks

We begin, for simplicity, with the case of scalar QCD, i.e., the case of a spin-0 quark (described by the scalar field  $\phi$ ) coupled to a non-Abelian gauge field  $A^\mu \equiv A^\mu_a T_a$ ,  $T_a$  ( $a = 1, \dots, N_c^2 - 1$ ) being the generators of the Lie algebra of the colour group  $SU(N_c)$ . We take  $\phi$  to stay in the vector space of the fundamental representation of  $SU(N_c)$ : i.e.,  $\phi$  stands for a vector  $\phi_i$  ( $i = 1, \dots, N_c$ ) in the colour space and  $T_a$  are the generators in the fundamental representation  $[(T_a)_{ij}, i, j \in \{1, \dots, N_c\}]$ . We limit ourselves to the case

of one single flavour. The unrenormalized Lagrangian is:

$$L(\phi, \phi^\dagger, A) = [D^\mu \phi]^\dagger D_\mu \phi - m_0^2 \phi^\dagger \phi - \frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu} , \quad (2.1)$$

where  $D^\mu = \partial^\mu + igA^\mu$  is the covariant derivative.

The unrenormalized scalar quark propagator will be denoted as

$$\langle T[\phi_i(x) \phi_j^\dagger(y)] \rangle = S_{ij}(x - y) . \quad (2.2)$$

Let us define the “physical” quark mass  $m$ , taken to be the pole mass, and the residue  $Z_W$  at the pole of the unrenormalized quark propagator by the following two equations:

$$[\tilde{S}(p)]^{-1} \big|_{p^2=m^2} = 0 \quad , \quad \tilde{S}_{ij}(p) \underset{p^2 \rightarrow m^2}{\simeq} \frac{iZ_W \delta_{ij}}{p^2 - m^2 + i\varepsilon} , \quad (2.3)$$

where  $\tilde{S}_{ij}(p)$  is the unrenormalized propagator in the momentum space:

$$\tilde{S}_{ij}(p) \equiv \int d^4z \, e^{ipz} S_{ij}(z) . \quad (2.4)$$

[We remind the reader that  $Z_W$  is not, in general, equal to the scalar-field renormalization constant  $Z_2$  (defined as  $\phi = Z_2^{1/2} \phi_R$ , where  $\phi$  is the bare field and  $\phi_R$  is the renormalized field); but it is equal to  $Z_2 \tilde{z}$ , where  $\tilde{z}$  is the residue at the pole of the renormalized propagator, defined as:

$$\tilde{S}_{Rij}(p) \underset{p^2 \rightarrow m^2}{\simeq} \frac{i\tilde{z} \delta_{ij}}{p^2 - m^2 + i\varepsilon} . \quad (2.5)$$

In fact one has that:

$$\tilde{S}_{ij}(p) = Z_2 \tilde{S}_{Rij}(p) \underset{p^2 \rightarrow m^2}{\simeq} \frac{iZ_2 \tilde{z} \delta_{ij}}{p^2 - m^2 + i\varepsilon} \equiv \frac{iZ_W \delta_{ij}}{p^2 - m^2 + i\varepsilon} , \quad (2.6)$$

that is,  $Z_W = Z_2 \tilde{z}$ , q.e.d. .]

In order to derive the scattering matrix elements following the LSZ approach [11], we need to know the on-shell *truncated-connected* Green functions, which are obtained from

the connected Green functions by removing the external legs calculated on-shell. We first consider the scattering of a quark in a given external gluon field  $A^\mu$ :

$$\phi(p, j) \rightarrow \phi(p', i) , \quad (2.7)$$

where  $i, j$  are colour indices ( $i, j = 1, \dots, N_c$ ). We define the truncated-connected propagator in the external gluon field  $A^\mu$ , in the momentum space as:

$$\tilde{S}_{ij}^{(tc)}(p, p'|A) \equiv \lim_{p^2, p'^2 \rightarrow m^2} \frac{p^2 - m^2}{i} \tilde{S}_{ij}(p, p'|A) \frac{p'^2 - m^2}{i} , \quad (2.8)$$

where  $m$  is the physical mass defined above and  $\tilde{S}_{ij}(p, p'|A)$  is the Fourier transform of  $S_{ij}(x, y|A)$ , the scalar propagator in an external gluon field, in the coordinate representation:

$$\tilde{S}_{ij}(p, p'|A) \equiv \int d^4x \int d^4y \exp[i(p'x - py)] S_{ij}(x, y|A) . \quad (2.9)$$

[Let us observe that, for convenience and simplicity reasons, we have not included a factor  $Z_W^{-2}$  in the definition (2.8) of the truncated-connected propagator. These omitted factors must be properly included when deriving the scattering amplitudes (see Sect. 5).] In the following we shall compute the truncated-connected propagator  $\tilde{S}_{ij}^{(tc)}(p, p'|A)$  in the so-called *eikonal* approximation, which is valid in the case of scattering particles with very high energy ( $E \equiv p^0 \simeq |\vec{p}| \gg m$ ) and small transferred momentum  $q \equiv p' - p$  (i.e.,  $\sqrt{|t|} \ll E$ , where  $t = q^2$ ). For example, if  $p^\mu \simeq p'^\mu \simeq (E, E, 0, 0)$ , one has that  $p_+ \simeq p'_+ \simeq 2E$  and  $p_- \simeq p'_- \simeq 0$ , where the following general notation has been used for a given four-vector  $V^\mu$ :

$$V_+ \equiv V^0 + V^1 , \quad V_- \equiv V^0 - V^1 . \quad (2.10)$$

We shall call  $V_+$  and  $V_-$  the “*longitudinal*” components of the four-vector  $V^\mu$ , while  $\vec{V}_\perp \equiv (V^2, V^3)$  is the component of  $V^\mu$  in the “*transverse*” plane ( $y, z$ ).

Our strategy consists in evaluating the truncated-connected propagator  $\tilde{S}_{ij}^{(tc)}(p, p'|A)$  in each order in perturbation theory considering  $L_\phi \equiv [D^\mu \phi]^\dagger D_\mu \phi - m_0^2 \phi^\dagger \phi = L_0 + L_{int}$ , where

$$L_0 = \partial^\mu \phi^\dagger \partial_\mu \phi - m^2 \phi^\dagger \phi \quad (2.11)$$

is the “free” (i.e., unperturbed) quark Lagrangian, which defines the “free” quark propagator  $i/(p^2 - m^2 + i\varepsilon)$ , with the physical mass  $m$ , and

$$L_{int} = -ig \left( \phi^\dagger \partial_\mu \phi - \partial_\mu \phi^\dagger \phi \right) A^\mu + g^2 \phi^\dagger \phi A^\mu A_\mu + \delta m^2 \phi^\dagger \phi \quad (2.12)$$

is the “interaction” Lagrangian, i.e., the “perturbation”. The squared-mass shift  $\delta m^2$  is defined as

$$\delta m^2 \equiv m^2 - m_0^2. \quad (2.13)$$

Let us start, therefore, by evaluating the  $n$ -th order term ( $n \geq 1$ ) in the perturbative expansion of the truncated-connected scalar propagator in an external gluon field  $A^\mu$ , in the eikonal approximation. This contribution, that we shall indicate as  $[\tilde{S}_{ij}^{(tc)}(p, p'|A)]_{(n)}$ , is schematically represented in Fig. 1a: only the quark-quark-gluon vertex [the first term appearing in  $L_{int}$  in Eq. (2.12)] contributes to the propagator in the eikonal limit that we are considering. Let us discuss in detail how this approximation is justified. The key-point (see also Ref. [4] and references therein) is that, in the high-energy limit we are considering,  $p \simeq p'$  and quarks retain their large longitudinal momenta during their scattering process. In other words, we are assuming that the external gluon field has a frequency distribution, i.e., Fourier transform,  $\tilde{A}_\mu(k)$  [defined by Eq. (2.15) below], such that the relevant phase-space region in  $d^4k$  is the one where  $k$  is negligible when compared to  $p$ . In Refs. [7, 8], the eikonal approximation has been done under the hypothesis that the external gluon field contains only a limited range of frequencies: in other words,  $A_\mu(x)$  is assumed to vary slowly on the scale set by the wavelength of the incoming waves. We shall come back to a detailed discussion on all these approximations and hypotheses in Sect. 5.

One must observe (see Table 1) that the scalar theory, in addition to the quark-quark-gluon vertex, has also two other types of vertices of the form  $g^2 \phi^\dagger \phi A^\mu A_\mu$  (quark-quark-gluon-gluon) and  $\delta m^2 \phi^\dagger \phi$  (squared-mass shift). Yet one can easily be convinced that the contributions due to these additional scalar vertices to the  $n$ -th order term in the perturbative expansion of the truncated-connected scalar propagator  $\tilde{S}_{ij}^{(tc)}(p, p'|A)$  in the high-energy limit are suppressed with respect to the contribution coming from the quark-quark-gluon couplings. This is essentially due to the fact that the quark-quark-gluon-gluon vertex and the squared-mass shift vertex do not carry momentum (see Table 1).

Table 1: The relevant vertices in scalar QCD.

<i>Vertex :</i>	<i>Feynman rule :</i>
$\phi_i^\dagger(p_2)\phi_j(p_1)A_a^\mu$	$-ig(p_1 + p_2)^\mu(T_a)_{ij}$
$\phi_i^\dagger\phi_j A_a^\mu A_b^\nu$	$ig^2 g^{\mu\nu}\{T_a, T_b\}_{ij}$
$\delta m^2 \phi_i^\dagger \phi_j$	$i\delta m^2 \delta_{ij}$

By virtue of the above-mentioned approximations, the expression for  $[\tilde{S}_{ij}^{(tc)}(p, p'|A)]_{(n)}$  reads as follows (see Fig. 1a):

$$\begin{aligned}
[\tilde{S}_{ij}^{(tc)}(p, p'|A)]_{(n)} &\simeq \\
&\simeq \int \frac{d^4 q_1}{(2\pi)^4} \cdots \int \frac{d^4 q_n}{(2\pi)^4} (2\pi)^4 \delta^{(4)}(q - q_1 - \cdots - q_n) \\
&\times \{ [-ig2p^{\mu_n} \tilde{A}_{\mu_n}(q_n)] \cdots [-ig2p^{\mu_1} \tilde{A}_{\mu_1}(q_1)] \}_{ij} \\
&\times \frac{i}{(p + q_1 + \cdots + q_{n-1})^2 - m^2 + i\varepsilon} \cdots \frac{i}{(p + q_1)^2 - m^2 + i\varepsilon} , \tag{2.14}
\end{aligned}$$

where  $q = p' - p$  is the transferred momentum. We have indicated with  $\tilde{A}_\mu(k)$  the Fourier transform at four-momentum  $k$  of the external gluon field  $A_\mu(z)$  in the coordinate representation:

$$\tilde{A}_\mu(k) \equiv \int d^4 z \, e^{ikz} A_\mu(z) . \tag{2.15}$$

In the eikonal limit, one also finds that:

$$(p + q_1 + \cdots + q_{n-1})^2 - m^2 \simeq 2E(q_{1-} + \cdots + q_{(n-1)-}) , \tag{2.16}$$

and similarly for the other expressions. Therefore, Eq. (2.14) becomes, using also Eq. (2.15) to express the external gluon field in the coordinate representation:

$$\begin{aligned}
[\tilde{S}_{ij}^{(tc)}(p, p'|A)]_{(n)} &\simeq \\
&\simeq \int d^4 b_n \cdots \int d^4 b_1 \, e^{iqb_n} \{ [-ig2p^{\mu_n} A_{\mu_n}(b_n)] \cdots [-ig2p^{\mu_1} A_{\mu_1}(b_1)] \}_{ij} \\
&\times \int \frac{d^4 q_1}{(2\pi)^4} \frac{i \, e^{iq_1(b_1 - b_n)}}{2E q_{1-} + i\varepsilon} \cdots \int \frac{d^4 q_{n-1}}{(2\pi)^4} \frac{i \, e^{iq_{n-1}(b_{n-1} - b_n)}}{2E(q_{1-} + \cdots + q_{(n-1)-}) + i\varepsilon} . \tag{2.17}
\end{aligned}$$

The last integration can be easily performed and one obtains:

$$\begin{aligned}
& [\tilde{S}_{ij}^{(tc)}(p, p'|A)]_{(n)} \simeq \\
& \simeq \int d^4 b_n \dots \int d^4 b_1 e^{iqb_n} \{[-ig2p^{\mu_n} A_{\mu_n}(b_n)] \dots [-ig2p^{\mu_1} A_{\mu_1}(b_1)]\}_{ij} \\
& \times \frac{1}{2E} \delta^{(2)}(\vec{b}_{(n-1)\perp} - \vec{b}_{n\perp}) \delta(b_{(n-1)-} - b_{n-}) \theta(b_{n+} - b_{(n-1)+}) \\
& \times \int \frac{d^4 q_1}{(2\pi)^4} \frac{i e^{iq_1(b_1 - b_{n-1})}}{2E q_{1-} + i\varepsilon} \dots \int \frac{d^4 q_{n-2}}{(2\pi)^4} \frac{i e^{iq_{n-2}(b_{n-2} - b_{n-1})}}{2E(q_{1-} + \dots + q_{(n-2)-}) + i\varepsilon} . \quad (2.18)
\end{aligned}$$

In the derivation of this result, we have used the following integral expression for the *step* function [ $\theta(\alpha) = 1$ , for  $\alpha > 0$ ;  $\theta(\alpha) = 0$ , for  $\alpha < 0$ ]:

$$\theta(\alpha) = \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{e^{i\alpha\omega}}{\omega - i\varepsilon} d\omega . \quad (2.19)$$

Therefore, if we proceed recursively, we obtain the following result:

$$\begin{aligned}
& [\tilde{S}_{ij}^{(tc)}(p, p'|A)]_{(n)} \simeq \\
& \simeq \int d^2 \vec{b}_{n\perp} \int db_{n-} \int db_{n+} \int db_{(n-1)+} \dots \int db_{1+} e^{iqb_n} \\
& \times \frac{1}{(2E)^{n-1}} \theta(b_{n+} - b_{(n-1)+}) \dots \theta(b_{2+} - b_{1+}) \\
& \times \{[-igp^{\mu_n} A_{\mu_n}(b_n)] \dots [-igp^{\mu_1} A_{\mu_1}(b_1)]\}_{ij} |_{b_{i-}=b_{n-} ; \vec{b}_{i\perp}=\vec{b}_{n\perp}} . \quad (2.20)
\end{aligned}$$

Obviously, we can parametrize the coordinates  $b_i$  ( $i = 1, \dots, n$ ) in the integral as follows:

$$b_i = b + p\tau_i \quad (i = 1, \dots, n) , \quad (2.21)$$

where  $\tau_i$  (or, better,  $\nu_i \equiv m\tau_i$ ) are proper-time variables and  $b$  is a four-vector with  $b_- = b_{n-}$ ,  $\vec{b}_\perp = \vec{b}_{n\perp}$  and a fixed  $b_+$ . In fact, using the fact that  $p \simeq (E, E, 0, 0)$ , i.e.,  $p_- \simeq 0$  and  $\vec{p}_\perp \simeq \vec{0}_\perp$ , we find that  $b_{i-} = b_-$  and  $\vec{b}_{i\perp} = \vec{b}_\perp$ ,  $\forall i = 1, \dots, n$ . Moreover,  $q = p' - p$ , so that  $q_+, q_- \simeq 0$  and  $qb_n \simeq -\vec{q}_\perp \cdot \vec{b}_{n\perp} = -\vec{q}_\perp \cdot \vec{b}_\perp \simeq qb$ . Therefore, using also the fact that  $db_{i+} = p_+ d\tau_i = 2E d\tau_i$ , we can re-write Eq. (2.20) as follows:

$$\begin{aligned}
& [\tilde{S}_{ij}^{(tc)}(p, p'|A)]_{(n)} \simeq \\
& \simeq 2E \int [d^3 b] e^{iqb} \int d\tau_1 \dots \int d\tau_n \theta(\tau_n - \tau_{n-1}) \dots \theta(\tau_2 - \tau_1) \\
& \times \{[-igp^{\mu_n} A_{\mu_n}(b + p\tau_n)] \dots [-igp^{\mu_1} A_{\mu_1}(b + p\tau_1)]\}_{ij} , \quad (2.22)
\end{aligned}$$



where we have used the notation:

$$[d^3b] \equiv d^2\vec{b}_\perp db_- . \quad (2.23)$$

As a general rule, in “ $[d^3b]$ ” one must not include the longitudinal component of  $b^\mu$  which is parallel to  $p^\mu$ . In other words, if  $p^\mu \simeq p'^\mu \simeq (E, E, 0, 0)$  (i.e.,  $p_+ \simeq 2E$ ,  $p_- \simeq 0$ ), one has that  $[d^3b] \equiv d^2\vec{b}_\perp db_-$ , while, if  $p^\mu \simeq p'^\mu \simeq (E, -E, 0, 0)$  (i.e.,  $p_+ \simeq 0$ ,  $p_- \simeq 2E$ ), then  $[d^3b] \equiv d^2\vec{b}_\perp db_+$ .

Eq. (2.22) is the  $n$ -th order term of  $\tilde{S}_{ij}^{(tc)}(p, p'|A)$ . Summing all orders ( $n \geq 1$ ), we finally obtain (see also Ref. [9]):

$$\begin{aligned} \tilde{S}_{ij}^{(tc)}(p, p'|A) &\simeq \\ &\simeq 2E \int [d^3b] e^{iqb} \left[ T \exp \left( -ig \int_{-\infty}^{+\infty} A_\mu(b + p\tau) p^\mu d\tau \right) - \mathbf{1} \right]_{ij} \\ &= 2E \int [d^3b] e^{iqb} [W_p(b) - \mathbf{1}]_{ij} , \end{aligned} \quad (2.24)$$

where  $W_p(b) = T \exp(\dots)$  is the *time*-ordered exponential, defined as:

$$\begin{aligned} W_p(b) &\equiv T \exp \left( -ig \int_{-\infty}^{+\infty} A_\mu(b + p\tau) p^\mu d\tau \right) \equiv \\ &\equiv \sum_{n=0}^{\infty} \int d\tau_1 \dots \int d\tau_n \theta(\tau_n - \tau_{n-1}) \dots \theta(\tau_2 - \tau_1) \\ &\quad \times [-igp^{\mu_n} A_{\mu_n}(b + p\tau_n)] \dots [-igp^{\mu_1} A_{\mu_1}(b + p\tau_1)] . \end{aligned} \quad (2.25)$$

Eq. (2.24) gives the expression for the truncated-connected scalar propagator in an external gluon field, in the eikonal approximation.

Proceeding exactly in the same way, we can also derive the following expression for the  $n$ -th order term ( $n \geq 1$ ) in the perturbative expansion of the truncated-connected propagator of a scalar antiquark in an external gluon field  $A^\mu$ ,  $\phi^\dagger(p, j) \rightarrow \phi^\dagger(p', i)$ , in the eikonal approximation:

$$\begin{aligned} [\tilde{S}^{(tc)}(\phi_{p,j}^\dagger \rightarrow \phi_{p',i}^\dagger | A)]_{(n)} &\simeq \\ &\simeq 2E \int [d^3b] e^{iqb} \int d\tau_1 \dots \int d\tau_n \theta(\tau_n - \tau_{n-1}) \dots \theta(\tau_2 - \tau_1) \\ &\quad \times \{ [igp^{\mu_1} A_{\mu_1}(b + p\tau_1)] \dots [igp^{\mu_n} A_{\mu_n}(b + p\tau_n)] \}_{ji} . \end{aligned} \quad (2.26)$$

And therefore, summing all orders:

$$\tilde{S}^{(tc)}(\phi_{p,j}^\dagger \rightarrow \phi_{p',i}^\dagger | A) \simeq 2E \int [d^3b] e^{iqb} \left[ \overline{T} \exp \left( ig \int_{-\infty}^{+\infty} A_\mu(b + p\tau) p^\mu d\tau \right) - \mathbf{1} \right]_{ji} . \quad (2.27)$$

We have denoted with  $\overline{T} \exp(\dots)$  the *antitime*-ordered exponential, defined as:

$$\begin{aligned} \overline{T} \exp \left( ig \int_{-\infty}^{+\infty} A_\mu(b + p\tau) p^\mu d\tau \right) &\equiv \\ &\equiv \sum_{n=0}^{\infty} \int d\tau_1 \dots \int d\tau_n \theta(\tau_n - \tau_{n-1}) \dots \theta(\tau_2 - \tau_1) \\ &\times [igp^{\mu_1} A_{\mu_1}(b + p\tau_1)] \dots [igp^{\mu_n} A_{\mu_n}(b + p\tau_n)] . \end{aligned} \quad (2.28)$$

Let us observe that in the *antitime* ordering the matrices  $A$  are ordered from left to right as they “appear” along the path going from  $\tau = -\infty$  to  $\tau = +\infty$  (i.e., when increasing the proper time). On the contrary, in the *time* ordering they are ordered from left to right as they “appear” along the path going from  $\tau = +\infty$  to  $\tau = -\infty$  (i.e., when decreasing the proper time).

Now, using the fact that  $A_\mu = A_\mu^a T^a$ , with  $(T^a)^\dagger = T^a$ , i.e.,  $(T^a)_{ji} = (T^{a*})_{ij}$ , we find:

$$[A_{\mu_1}(b + p\tau_1) \dots A_{\mu_n}(b + p\tau_n)]_{ji} = [A_{\mu_n}^*(b + p\tau_n) \dots A_{\mu_1}^*(b + p\tau_1)]_{ij} . \quad (2.29)$$

Therefore, we can write Eq. (2.27) as follows:

$$\begin{aligned} \tilde{S}^{(tc)}(\phi_{p,j}^\dagger \rightarrow \phi_{p',i}^\dagger | A) &\simeq \\ &\simeq 2E \int [d^3b] e^{iqb} \left[ T \exp \left( ig \int_{-\infty}^{+\infty} A_\mu^*(b + p\tau) p^\mu d\tau \right) - \mathbf{1} \right]_{ij} \\ &\simeq 2E \int [d^3b] e^{iqb} [W_p^*(b) - \mathbf{1}]_{ij} , \end{aligned} \quad (2.30)$$

where  $W_p(b)$  has been defined in Eq. (2.25).

When comparing with the result (2.24), we see that the scattering amplitude of an antiquark in the external gluon field  $A_\mu$  is equal to the scattering amplitude of a quark in the charge-conjugated (C-transformed) gluon field  $A'_\mu = -A_\mu^t = -A_\mu^*$ , as expected.

### 3. The scattering of quarks and antiquarks

We want now to extend the results obtained in the previous section to the case (more interesting from the physical point of view) of “real” fermion QCD: that is a spin-1/2 quark coupled to a non-Abelian gauge field.

As before, we limit ourselves to the case of one single flavour. The unrenormalized QCD Lagrangian is:

$$L(\psi, \psi^\dagger, A) = \bar{\psi}(i\gamma^\mu D_\mu - m_0)\psi - \frac{1}{4}F_{\mu\nu}^a F^{a\mu\nu} , \quad (3.1)$$

where  $D^\mu = \partial^\mu + igA^\mu$  is the covariant derivative [ $A^\mu \equiv A_a^\mu T_a$ ,  $T_a$  ( $a = 1, \dots, N_c^2 - 1$ ) being the generators of the Lie algebra of the colour group  $SU(N_c)$ ] and  $\psi$  stands for a vector  $\psi_i$  ( $i = 1, \dots, N_c$ ) in the colour vector space of the fundamental representation.

The unrenormalized quark propagator will be denoted as

$$\langle T[\psi_i(x)\bar{\psi}_j(y)] \rangle = G_{ij}(x - y) . \quad (3.2)$$

Let us define the “physical” quark mass  $m$ , taken to be the pole mass, and the residue  $Z_W$  at the pole of the unrenormalized quark propagator by the following two equations:

$$[\tilde{G}(p)]^{-1}|_{p^2=m^2} = 0 \quad , \quad \tilde{G}_{ij}(p) \underset{p^2 \rightarrow m^2}{\simeq} \frac{iZ_W \delta_{ij}}{\hat{p} - m + i\varepsilon} , \quad (3.3)$$

where  $\tilde{G}_{ij}(p)$  is the unrenormalized propagator in the momentum space:

$$\tilde{G}_{ij}(p) \equiv \int d^4z \, e^{ipz} G_{ij}(z) . \quad (3.4)$$

In Eq. (3.3) we have used the notation:  $\hat{a} \equiv \gamma^\mu a_\mu$ .

As in the previous section, we consider the scattering of a quark in a given external gluon field  $A^\mu$ . The truncated-connected fermion propagator in the momentum space is defined as:

$$\tilde{G}_{ij}^{(tc)}(p, p'|A) \equiv \lim_{p^2, p'^2 \rightarrow m^2} \frac{\hat{p}' - m}{i} \tilde{G}_{ij}(p, p'|A) \frac{\hat{p} - m}{i} , \quad (3.5)$$

where  $m$  is the physical quark mass defined above and  $\tilde{G}_{ij}(p, p'|A)$  is the Fourier transform [see Eq. (2.9)] of  $G_{ij}(x, y|A)$ , the truncated–connected fermion propagator in the coordinate representation. The matrix element for the scattering of a quark in a given external gluon field  $A^\mu$ ,

$$\psi(p, j, \beta) \rightarrow \psi(p', i, \alpha) , \quad (3.6)$$

where  $i, j$  are colour indices and  $\alpha, \beta$  are spin indices, is given by  $\bar{u}_\alpha(p')\tilde{G}_{ij}^{(tc)}(p, p'|A)u_\beta(p)$ , where  $u_\alpha(p)$  are the “positive–energy” spinors  $[(\hat{p} - m)u_\alpha(p) = \bar{u}_\alpha(p)(\hat{p} - m) = 0]$  with the usual relativistic normalization:

$$\bar{u}_\alpha(p)\gamma^\mu u_\beta(p) = 2p^\mu\delta_{\alpha\beta} \quad , \quad \bar{u}_\alpha(p)u_\beta(p) = 2m\delta_{\alpha\beta} . \quad (3.7)$$

In the high–energy limit we are considering, we can make the following replacement:

$$\bar{u}_\alpha(p')\tilde{G}_{ij}^{(tc)}(p, p'|A)u_\beta(p) \simeq \delta_{\alpha\beta} \cdot \tilde{S}_{ij}^{(tc)}(p, p'|A) , \quad (3.8)$$

where  $\tilde{S}_{ij}^{(tc)}(p, p'|A)$  is the truncated–connected propagator for a scalar (i.e., spin–0) quark in the external gluon field  $A^\mu$ , which was discussed in the previous section. Let us see how this approximation is justified. As in the scalar case, our strategy consists in evaluating the truncated–connected propagator  $\tilde{G}_{ij}^{(tc)}(p, p'|A)$ , or better the quantity (3.8), in each order in perturbation theory, considering  $L_\psi \equiv \bar{\psi}(i\gamma^\mu D_\mu - m_0)\psi = L_0 + L_{int}$ , where

$$L_0 = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi \quad (3.9)$$

is the “free” (i.e., “unperturbed”) quark Lagrangian, which defines the “free” quark propagator  $i/(\hat{p} - m + i\varepsilon)$ , with the physical mass  $m$ , and

$$L_{int} = -g\bar{\psi}\gamma^\mu A_\mu\psi + \delta m\bar{\psi}\psi \quad (3.10)$$

is the “interaction” Lagrangian, i.e., the “perturbation”. The mass shift  $\delta m$  is defined as:

$$\delta m \equiv m - m_0 . \quad (3.11)$$

Using the fact that  $\bar{u}(p')(\hat{p}' - m)u(p) = \bar{u}(p')(\hat{p} - m)u(p) = 0$  and the definition (3.5) of the truncated–connected fermion propagator, one can easily derive the following expression

for the  $n$ -th order term in the perturbative expansion ( $n \geq 1$ ):

$$\begin{aligned}
& [\bar{u}_\alpha(p') \tilde{G}_{ij}^{(tc)}(p, p'|A) u_\beta(p)]_{(n)} = \\
& = \int \frac{d^4 q_1}{(2\pi)^4} \cdots \int \frac{d^4 q_n}{(2\pi)^4} (2\pi)^4 \delta^{(4)}(q - q_1 - \dots - q_n) N_{\alpha\beta, ij}^{(ferm)}(q_1, \dots, q_n) \\
& \times \frac{i}{(p + q_1 + \dots + q_{n-1})^2 - m^2 + i\varepsilon} \cdots \frac{i}{(p + q_1)^2 - m^2 + i\varepsilon} , \tag{3.12}
\end{aligned}$$

where  $N_{\alpha\beta, ij}^{(ferm)}(q_1, \dots, q_n)$  is given by:

$$\begin{aligned}
& N_{\alpha\beta, ij}^{(ferm)}(q_1, \dots, q_n) \equiv \\
& \equiv \bar{u}_\alpha(p') \{ [-ig\gamma^{\mu_n} \tilde{A}_{\mu_n}(q_n) + i\delta m (2\pi)^4 \delta^{(4)}(q_n) \cdot \mathbf{1}] (\hat{p} + \hat{q}_1 + \dots + \hat{q}_{n-1} + m) \\
& \dots (\hat{p} + \hat{q}_1 + m) [-ig\gamma^{\mu_1} \tilde{A}_{\mu_1}(q_1) + i\delta m (2\pi)^4 \delta^{(4)}(q_1) \cdot \mathbf{1}] \}_{ij} u_\beta(p) . \tag{3.13}
\end{aligned}$$

By virtue of the eikonal approximation, the relevant phase-space region in the  $n$ -th order term of the perturbative expansion (3.12) has the property that  $q_i$ ,  $m$  and  $\delta m$  are negligible when compared to  $p$  in the numerator  $N_{\alpha\beta, ij}^{(ferm)}(q_1, \dots, q_n)$ . Therefore, the numerator  $N_{\alpha\beta, ij}^{(ferm)}(q_1, \dots, q_n)$  in Eq. (3.12) can be approximated as:

$$N_{\alpha\beta, ij}^{(ferm)}(q_1, \dots, q_n) \simeq \{ -ig2p^{\mu_n} \tilde{A}_{\mu_n}(q_n) \cdot N_{\alpha\beta}^{(ferm)}(q_1, \dots, q_{n-1}) \}_{ij} . \tag{3.14}$$

Using the fact that:

$$\begin{aligned}
N_{\alpha\beta, ij}^{(ferm)}(q_1) &= \bar{u}_\alpha(p') \{ -ig\gamma^{\mu_1} \tilde{A}_{\mu_1}(q_1) + i\delta m (2\pi)^4 \delta^{(4)}(q_1) \cdot \mathbf{1} \}_{ij} u_\beta(p) \\
&\simeq \delta_{\alpha\beta} \cdot [-ig2p^{\mu_1} \tilde{A}_{\mu_1}(q_1)]_{ij} , \tag{3.15}
\end{aligned}$$

we find, proceeding recursively from Eq. (3.14):

$$N_{\alpha\beta, ij}^{(ferm)}(q_1, \dots, q_n) \simeq \delta_{\alpha\beta} \cdot \{ [-ig2p^{\mu_n} \tilde{A}_{\mu_n}(q_n)] \dots [-ig2p^{\mu_1} \tilde{A}_{\mu_1}(q_1)] \}_{ij} . \tag{3.16}$$

Apart from the *delta* function in front, which simply reflects the fact that fermions retain their helicities during the scattering process in the high-energy limit, this is exactly the term we would have expected at the numerator of the  $n$ -th order term in the perturbative expansion of the truncated-connected scalar propagator  $\tilde{S}_{ij}^{(tc)}(p, p'|A)$  in the high-energy limit [the denominators in (3.12) are already equal to the scalar case!]. In fact, as reported in Table 1, the factors  $(-ig2p^\mu)$  in (2.14) come from the quark-quark-gluon vertex of the

scalar theory in the high-energy limit (when  $p \simeq p'$ ). From Eqs. (3.12), (3.16) and (2.14) we derive

$$[\bar{u}_\alpha(p') \tilde{G}_{ij}^{(tc)}(p, p'|A) u_\beta(p)]_{(n)} \simeq \delta_{\alpha\beta} \cdot [\tilde{S}_{ij}^{(tc)}(p, p'|A)]_{(n)} , \quad (3.17)$$

for each order  $n$  in the perturbative expansion, so proving Eq. (3.8). Therefore, using the result (2.24) derived in the previous section, we find the following expression for the quantity (3.8) in the eikonal approximation (see also Ref. [9]):

$$\begin{aligned} \bar{u}_\alpha(p') \tilde{G}_{ij}^{(tc)}(p, p'|A) u_\beta(p) &\simeq \delta_{\alpha\beta} \cdot \tilde{S}_{ij}^{(tc)}(p, p'|A) \\ &\simeq \delta_{\alpha\beta} \cdot 2E \int [d^3b] e^{iqb} \left[ T \exp \left( -ig \int_{-\infty}^{+\infty} A_\mu(b + p\tau) p^\mu d\tau \right) - \mathbf{1} \right]_{ij} \\ &= \delta_{\alpha\beta} \cdot 2E \int [d^3b] e^{iqb} [W_p(b) - \mathbf{1}]_{ij} , \end{aligned} \quad (3.18)$$

where  $W_p(b)$  is the time-ordered Wilson string along the path  $x(\tau) = b + p\tau$ ,  $\tau \in [-\infty, +\infty]$ , defined in Eq. (2.25).

Proceeding exactly in the same way, we can also derive the following expression for the truncated-connected propagator of an antiquark in an external gluon field  $A^\mu$ ,  $\bar{\psi}(p, j, \beta) \rightarrow \bar{\psi}(p', i, \alpha)$ , in the eikonal approximation:

$$\begin{aligned} \bar{v}_\beta(p) \tilde{G}^{(tc)}(\bar{\psi}_{p,j} \rightarrow \bar{\psi}_{p',i}|A) v_\alpha(p') &\simeq \delta_{\alpha\beta} \cdot \tilde{S}^{(tc)}(\phi_{p,j}^\dagger \rightarrow \phi_{p',i}^\dagger|A) \\ &\simeq \delta_{\alpha\beta} \cdot 2E \int [d^3b] e^{iqb} \left[ T \exp \left( ig \int_{-\infty}^{+\infty} A_\mu^*(b + p\tau) p^\mu d\tau \right) - \mathbf{1} \right]_{ij} \\ &= \delta_{\alpha\beta} \cdot 2E \int [d^3b] e^{iqb} [W_p^*(b) - \mathbf{1}]_{ij} ; \end{aligned} \quad (3.19)$$

$v_\alpha(p)$  are the “negative-energy” spinors  $[(\hat{p} + m)v_\alpha(p) = \bar{v}_\alpha(p)(\hat{p} + m) = 0]$  with the usual relativistic normalization:

$$\bar{v}_\alpha(p) \gamma^\mu v_\beta(p) = 2p^\mu \delta_{\alpha\beta} \quad , \quad \bar{v}_\alpha(p) v_\beta(p) = -2m \delta_{\alpha\beta} . \quad (3.20)$$

In fact, one can easily derive the following expression for the  $n$ -th order term in the perturbative expansion ( $n \geq 1$ ):

$$\begin{aligned} &[\bar{v}_\beta(p) \tilde{G}^{(tc)}(\bar{\psi}_{p,j} \rightarrow \bar{\psi}_{p',i}|A) v_\alpha(p')]_{(n)} = \\ &= \int \frac{d^4q_1}{(2\pi)^4} \cdots \int \frac{d^4q_n}{(2\pi)^4} (2\pi)^4 \delta^{(4)}(q - q_1 - \dots - q_n) \bar{N}_{\alpha\beta, ij}^{(ferm)}(q_1, \dots, q_n) \\ &\times \frac{i}{(p + q_1)^2 - m^2 + i\varepsilon} \cdots \frac{i}{(p + q_1 + \dots + q_{n-1})^2 - m^2 + i\varepsilon} , \end{aligned} \quad (3.21)$$

where  $\overline{N}_{\alpha\beta, ij}^{(ferm)}(q_1, \dots, q_n)$  is given by:

$$\begin{aligned} \overline{N}_{\alpha\beta, ij}^{(ferm)}(q_1, \dots, q_n) &\equiv \\ &\equiv \overline{v}_\beta(p) \{ [ig\gamma^{\mu_1} \tilde{A}_{\mu_1}(q_1) - i\delta m (2\pi)^4 \delta^{(4)}(q_1) \cdot \mathbf{1}] (\hat{p} + \hat{q}_1 - m) \dots \\ &\quad (\hat{p} + \hat{q}_1 + \dots + \hat{q}_{n-1} - m) [ig\gamma^{\mu_n} \tilde{A}_{\mu_n}(q_n) - i\delta m (2\pi)^4 \delta^{(4)}(q_n) \cdot \mathbf{1}] \}_{ji} v_\alpha(p') . \end{aligned} \quad (3.22)$$

Proceeding as for the derivation of Eq. (3.16) and making use of Eq. (2.29), we find:

$$\begin{aligned} \overline{N}_{\alpha\beta, ij}^{(ferm)}(q_1, \dots, q_n) &\simeq \\ &\simeq \delta_{\alpha\beta} \cdot \{ [ig2p^{\mu_1} \tilde{A}_{\mu_1}(q_1)] \dots [ig2p^{\mu_n} \tilde{A}_{\mu_n}(q_n)] \}_{ji} \\ &= \delta_{\alpha\beta} \cdot \{ [ig2p^{\mu_n} \tilde{A}_{\mu_n}^*(q_n)] \dots [ig2p^{\mu_1} \tilde{A}_{\mu_1}^*(q_1)] \}_{ij} . \end{aligned} \quad (3.23)$$

Apart from the *delta* function in front, this is exactly the term we would have expected at the numerator of the  $n$ -th order term in the perturbative expansion of the truncated-connected scalar antiquark propagator  $\tilde{S}^{(tc)}(\phi_{p,j}^\dagger \rightarrow \phi_{p',i}^\dagger | A)$  in the high-energy limit. From Eqs. (3.21) and (3.23) we derive

$$[\overline{v}_\beta(p) \tilde{G}^{(tc)}(\overline{\psi}_{p,j} \rightarrow \overline{\psi}_{p',i} | A) v_\alpha(p')]_{(n)} \simeq \delta_{\alpha\beta} \cdot [\tilde{S}^{(tc)}(\phi_{p,j}^\dagger \rightarrow \phi_{p',i}^\dagger | A)]_{(n)} , \quad (3.24)$$

for every order  $n$  in the perturbative expansion, so proving Eq. (3.19). As for the scalar case, going from quarks to antiquarks corresponds just to the change from the fundamental representation  $T_a$  of  $SU(N_c)$  to the complex conjugate representation  $T'_a = -T_a^*$ .

## 4. The scattering of gluons

By using the same techniques developed in the previous sections, we shall evaluate the “truncated-connected gluon propagator” in a given external gluon field  $A_b^\nu$ , in the eikonal approximation. We call this quantity “ $\tilde{D}_{\mu'\mu, a'a}^{(tc)}(k, k' | A)$ ”, whose  $n$ -th order perturbative term is defined schematically in Fig. 1b, where the external legs are supposed to be truncated on-shell ( $k^2, k'^2 \rightarrow 0$ ). More precisely, we shall evaluate the following quantity:

$$\varepsilon_{(\lambda')}^{\mu'*}(k') \tilde{D}_{\mu'\mu, a'a}^{(tc)}(k, k' | A) \varepsilon_{(\lambda)}^\mu(k) , \quad (4.1)$$

where  $\varepsilon_{(\lambda)}^\mu(k)$  are the polarization four-vectors ( $\lambda, \lambda' \in \{1, 2\}$ ):

$$\varepsilon_{(\lambda)}(k) \cdot \varepsilon_{(\lambda')}^*(k) = -\delta_{\lambda\lambda'} \quad , \quad k \cdot \varepsilon_{(\lambda)}(k)|_{k^2=0} = 0 \quad . \quad (4.2)$$

This quantity should describe (under certain approximations that will be discussed below and also in the next section) the scattering matrix element of a gluon in a given external gluon field  $A_b^\nu$ :

$$g(k, a, \lambda) \rightarrow g(k', a', \lambda') \quad ; \quad (4.3)$$

$a, a' \in \{1, \dots, N_c^2 - 1\}$  are colour indices and  $\lambda, \lambda' \in \{1, 2\}$  are spin indices. In the eikonal approximation the dominant interaction between the incident gluon and the external gluon field is represented by the three-gluon vertex, which is linear in the four-momentum of the gluon (while the four-gluon vertex is not dependent on the momentum). Its Feynman rule is given by:

$$\begin{aligned} V_{\mu_1\mu_2\mu_3}^{a_1a_2a_3}(k_1, k_2, k_3) = \\ = gf^{a_1a_2a_3} [g_{\mu_1\mu_2}(-k_1 + k_2)_{\mu_3} + g_{\mu_2\mu_3}(-k_2 - k_3)_{\mu_1} + g_{\mu_3\mu_1}(k_3 + k_1)_{\mu_2}] \quad , \end{aligned} \quad (4.4)$$

where the four-momenta  $k_1$  and  $k_2$  are taken to be flowing into the vertex, while the four-momentum  $k_3$  is taken to be flowing out from the vertex. The explicit expression of the  $n$ -th order perturbative term of the quantity (4.1), which is schematically defined in Fig. 1b, is given by:

$$\begin{aligned} \left[ \varepsilon_{(\lambda')}^{\mu'*}(k') \tilde{D}_{\mu'\mu, a'a}^{(tc)}(k, k'|A) \varepsilon_{(\lambda)}^\mu(k) \right]_{(n)} \simeq \\ \simeq \int \frac{d^4q_1}{(2\pi)^4} \dots \int \frac{d^4q_n}{(2\pi)^4} (2\pi)^4 \delta^{(4)}(q - q_1 - \dots - q_n) N_{\lambda'\lambda, a'a}^{(gluon)}(q_1, \dots, q_n) \\ \times \frac{i}{(k + q_1 + \dots + q_{n-1})^2 + i\varepsilon} \dots \frac{i}{(k + q_1)^2 + i\varepsilon} \quad , \end{aligned} \quad (4.5)$$

where  $q \equiv k' - k$  is the transferred momentum and  $N_{\lambda'\lambda, a'a}^{(gluon)}(q_1, \dots, q_n)$  is given by:

$$\begin{aligned} N_{\lambda'\lambda, a'a}^{(gluon)}(q_1, \dots, q_n) \equiv \\ \equiv \varepsilon_{(\lambda)}^\mu(k) V_{\mu\nu_1\rho_1}^{ab_1c_1}(k, q_1, k + q_1) \tilde{A}_{b_1}^{\nu_1}(q_1) g^{\rho_1\rho_2} \delta_{c_1c_2} V_{\rho_2\nu_2\rho_3}^{c_2b_2c_3}(k + q_1, q_2, k + q_1 + q_2) \times \dots \\ \dots \times V_{\rho_{2n-2}\nu_n\mu'}^{c_{2n-2}b_na'}(k + q_1 + \dots + q_{n-1}, q_n, k') \tilde{A}_{b_n}^{\nu_n}(q_n) \varepsilon_{(\lambda')}^{\mu'*}(k') \quad . \end{aligned} \quad (4.6)$$

It is not difficult to verify that, in the eikonal approximation, using the properties (4.2) for the polarization four-vectors, the quantity (4.6) simplifies as follows:

$$N_{\lambda'\lambda, a'a}^{(gluon)}(q_1, \dots, q_n) \simeq \delta_{\lambda'\lambda} \cdot \{ [-ig2k^{\mu_n} \tilde{\mathcal{A}}_{\mu_n}(q_n)] \dots [-ig2k^{\mu_1} \tilde{\mathcal{A}}_{\mu_1}(q_1)] \}_{a'a} \quad , \quad (4.7)$$



where we have used the notation:

$$\mathcal{A}_\mu \equiv A_\mu^b T_{(adj)}^b, \quad (4.8)$$

$T_{(adj)}^b$  being the  $N_c^2 - 1$  matrices of the  $SU(N_c)$  Lie algebra in the adjoint representation:

$$\left(T_{(adj)}^a\right)_{bc} = -if^{abc}. \quad (4.9)$$

The expression (4.5), with the result (4.7), is perfectly analogous to the corresponding expression appearing in Eq. (2.14) for the case of the scalar quark. By proceeding exactly as in Sect. 2, we can thus further approximate the above-written Eq. (4.5) as follows:

$$\begin{aligned} & \left[ \varepsilon_{(\lambda')}^{\mu'*}(k') \tilde{D}_{\mu'\mu, a'a}^{(tc)}(k, k'|A) \varepsilon_{(\lambda)}^\mu(k) \right]_{(n)} \simeq \\ & \simeq \delta_{\lambda'\lambda} \cdot 2E \int [d^3b] e^{iqb} \int d\tau_1 \dots \int d\tau_n \theta(\tau_n - \tau_{n-1}) \dots \theta(\tau_2 - \tau_1) \\ & \times \{ [-igk^{\mu_n} \mathcal{A}_{\mu_n}(b + k\tau_n)] \dots [-igk^{\mu_1} \mathcal{A}_{\mu_1}(b + k\tau_1)] \}_{a'a}. \end{aligned} \quad (4.10)$$

Therefore, summing all orders ( $n \geq 1$ ), we finally obtain:

$$\begin{aligned} & \varepsilon_{(\lambda')}^{\mu'*}(k') \tilde{D}_{\mu'\mu, a'a}^{(tc)}(k, k'|A) \varepsilon_{(\lambda)}^\mu(k) \simeq \\ & \simeq \delta_{\lambda'\lambda} \cdot 2E \int [d^3b] e^{iqb} [\mathcal{V}_k(b) - \mathbf{1}]_{a'a}, \end{aligned} \quad (4.11)$$

where  $\mathcal{V}_k(b)$  is the Wilson string along the path  $x(\tau) = b + k\tau$  ( $\tau \in [-\infty, +\infty]$ ), in the adjoint representation, defined as:

$$\begin{aligned} \mathcal{V}_k(b) & \equiv T \exp \left( -ig \int_{-\infty}^{+\infty} \mathcal{A}_\mu(b + k\tau) k^\mu d\tau \right) \equiv \\ & \equiv \sum_{n=0}^{\infty} \int d\tau_1 \dots \int d\tau_n \theta(\tau_n - \tau_{n-1}) \dots \theta(\tau_2 - \tau_1) \\ & \times [-igk^{\mu_n} \mathcal{A}_{\mu_n}(b + k\tau_n)] \dots [-igk^{\mu_1} \mathcal{A}_{\mu_1}(b + k\tau_1)]. \end{aligned} \quad (4.12)$$

Eq. (4.11) gives the expression for the scattering matrix element of a gluon in a given external gluon field, in the eikonal approximation.

## 5. Parton-parton scattering amplitudes

In the previous sections, we have derived expressions for the truncated-connected quark (antiquark) propagator in a given external gluon field  $A^\mu$ , by a direct resummation of perturbation theory in the limit of very high energy and small transferred momentum. The procedure has been also extended to include the case in which the scattering parton is a gluon: additional approximations are necessary and they will be discussed below.

The truncated-connected propagators in an external gluon field are the basic ingredients to derive high-energy parton-parton scattering amplitudes, using the LSZ reduction formulae and a functional integral approach. This was done in Ref. [9] and it will be quickly reviewed here for the convenience of the reader. Let us consider, for example, the elastic scattering process of two scalar quarks with initial four-momenta  $p_1$  and  $p_2$  and final four-momenta  $p'_1$  and  $p'_2$ :

$$\phi_j(p_1) + \phi_l(p_2) \rightarrow \phi_i(p'_1) + \phi_k(p'_2) . \quad (5.1)$$

[ $i, j, k, l \in \{1, \dots, N_c\}$  are colour indices.] In the center-of-mass reference system (c.m.s.), taking the initial trajectories of the two quarks along the  $x^1$ -axis, the four-momenta  $p_1$ ,  $p_2$ ,  $p'_1$  and  $p'_2$  are given, in the limit of “soft” high-energy scattering,  $s = (p_1 + p_2)^2 \rightarrow \infty$  and  $t = (p_1 - p'_1)^2 \ll s$ , by:

$$p_1 \simeq p'_1 \simeq (E, E, \vec{0}_t) \quad , \quad p_2 \simeq p'_2 \simeq (E, -E, \vec{0}_t) . \quad (5.2)$$

Using the LSZ reduction formulae and a functional integral approach, one finds the following expression for the scattering matrix element [with the plane wave functions normalized as:  $\phi_p(x) = \exp(-ipx)$ ]:

$$\begin{aligned} & \langle \phi_i(p'_1) \phi_k(p'_2) | (S - \mathbf{1}) | \phi_j(p_1) \phi_l(p_2) \rangle \simeq \\ & \simeq \frac{1}{Z_W^2} \{ \langle \tilde{S}_{ij}^{(tc)}(p_1, p'_1 | A) \tilde{S}_{kl}^{(tc)}(p_2, p'_2 | A) \rangle_A + \langle \tilde{S}_{kj}^{(tc)}(p_1, p'_2 | A) \tilde{S}_{il}^{(tc)}(p_2, p'_1 | A) \rangle_A \} , \end{aligned} \quad (5.3)$$

where  $Z_W$  is the residue at the pole (i.e., for  $p^2 \rightarrow m^2$ ) of the unrenormalized quark propagator [see Eq. (2.3)]. The expectation value  $\langle O(A) \rangle_A$  of an arbitrary functional  $O(A)$  of the gluon field  $A^\mu$  is defined as:

$$\langle O(A) \rangle_A \equiv \frac{1}{Z_{QCD}} \int [dA] O(A) \exp \left[ -\frac{i}{4} \int d^4x F_a^{\mu\nu} F_{a\mu\nu} \right] \{ \det[D^\mu D_\mu + m^2] \}^{-1} , \quad (5.4)$$

where  $Z_{QCD}$  is the partition function for scalar QCD:

$$\begin{aligned} Z_{QCD} &\equiv \int [dA][d\phi][d\phi^\dagger] \exp \left[ i \int d^4x \, L(\phi, \phi^\dagger, A) \right] \\ &= \int [dA] \exp \left[ -\frac{i}{4} \int d^4x \, F_a^{\mu\nu} F_{a\mu\nu} \right] \{ \det[D^\mu D_\mu + m^2] \}^{-1} . \end{aligned} \quad (5.5)$$

The determinant in Eqs. (5.4) and (5.5) comes from the integration over the scalar degrees of freedom. In fact, the Lagrangian  $L(\phi, \phi^\dagger, A)$  is bilinear in the scalar fields  $\phi$  and  $\phi^\dagger$ :

$$\int d^4x \left[ (D^\mu \phi)^\dagger D_\mu \phi - m^2 \phi^\dagger \phi \right] = - \int d^4x \, \phi^\dagger \left[ D^\mu D_\mu + m^2 \right] \phi . \quad (5.6)$$

Therefore, the functional integral over the scalar fields  $\phi$  and  $\phi^\dagger$  is an ordinary Gaussian integral and can be performed in the standard way, to give (apart from an irrelevant constant)  $\{ \det[D^\mu D_\mu + m^2] \}^{-1}$ .

The first expectation value in Eq. (5.3) corresponds to the  $t$ -channel scattering of the two particles, with squared transferred momentum equal to:  $(p_1 - p'_1)^2 \equiv t \ll s$ . The second expectation value corresponds instead to the  $u$ -channel scattering of the two quarks. In other words, the squared transferred momentum, flowing from one quark to the other, is equal to:

$$(p_1 - p'_2)^2 \equiv u = 4m^2 - s - t \simeq -s , \quad (5.7)$$

in the limit  $s \rightarrow \infty$  with  $t, m^2 \ll s$ . In this high-energy limit the contribution coming from the second expectation value in Eq. (5.3) is smaller by at least a factor of  $s$ , when compared with the first expectation value, and hence is negligible. [One can be easily convinced of this by considering the Feynman diagrams of the process in the perturbation theory: the diagrams which correspond to the second piece in Eq. (5.3) have intermediate gluons carrying a big squared transferred momentum  $u \simeq -s$ , so that their propagators suppress the corresponding amplitude.] Therefore, in our limit:

$$\langle \phi_i(p'_1) \phi_k(p'_2) | (S - \mathbf{1}) | \phi_j(p_1) \phi_l(p_2) \rangle \simeq \frac{1}{Z_W^2} \langle \tilde{S}_{ij}^{(tc)}(p_1, p'_1 | A) \tilde{S}_{kl}^{(tc)}(p_2, p'_2 | A) \rangle_A , \quad (5.8)$$

where for the truncated-connected propagator  $\tilde{S}^{(tc)}(p, p' | A)$  we can use the expression (2.24) derived in Sect. 2 in the eikonal limit. Also the quantity  $Z_W$  can be evaluated in the same approximation: the result and the details of the calculations are reported in the

Appendix. If one defines the diffusion amplitude  $T_{fi} = \langle f|T|i \rangle$  by

$$\begin{aligned} & \langle \phi_i(p'_1) \phi_k(p'_2) | (S - \mathbf{1}) | \phi_j(p_1) \phi_l(p_2) \rangle = \\ & = i(2\pi)^4 \delta^{(4)}(P_{fin} - P_{in}) \langle \phi_i(p'_1) \phi_k(p'_2) | T | \phi_j(p_1) \phi_l(p_2) \rangle , \end{aligned} \quad (5.9)$$

where  $P_{in} = p_1 + p_2$  is the initial total four-momentum and  $P_{fin} = p'_1 + p'_2$  is the final total four-momentum, the following results is obtained at the end [9]:

$$\begin{aligned} & \langle \phi_i(p'_1) \phi_k(p'_2) | T | \phi_j(p_1) \phi_l(p_2) \rangle \simeq \\ & \simeq -\frac{i}{Z_W^2} 2s \int d^2 \vec{z}_\perp e^{i\vec{q}_\perp \cdot \vec{z}_\perp} \langle [W_{p_1}(z_t) - \mathbf{1}]_{ij} [W_{p_2}(0) - \mathbf{1}]_{kl} \rangle_A , \end{aligned} \quad (5.10)$$

where  $q = p_1 - p'_1 \simeq (0, 0, \vec{q}_\perp)$ , with  $t = q^2 = -\vec{q}_\perp^2$ , is the transferred four-momentum, and  $z_t = (0, 0, \vec{z}_\perp)$  is the distance between the two trajectories in the transverse plane.

One can proceed exactly in the same way when considering the elastic scattering process of two “real” (i.e., spin-1/2) quarks with initial four-momenta  $p_1, p_2$  and final four-momenta  $p'_1, p'_2$ , in the limit of “soft” high-energy scattering:

$$\psi_{j\beta}(p_1) + \psi_{l\delta}(p_2) \rightarrow \psi_{i\alpha}(p'_1) + \psi_{k\gamma}(p'_2) . \quad (5.11)$$

The following expression for the scattering matrix element is derived using the LSZ reduction formulae and a functional integral approach [with the plane wave functions normalized as:  $\psi_p(x) = u(p) \exp(-ipx)$ , with  $\bar{u}_\alpha(p)u_\beta(p) = 2m\delta_{\alpha\beta}$ ]:

$$\begin{aligned} & \langle \psi_{i\alpha}(p'_1) \psi_{k\gamma}(p'_2) | (S - \mathbf{1}) | \psi_{j\beta}(p_1) \psi_{l\delta}(p_2) \rangle \simeq \\ & \simeq \frac{1}{Z_W^2} \langle \bar{u}_\alpha(p'_1) \tilde{G}_{ij}^{(tc)}(p_1, p'_1 | A) u_\beta(p_1) \cdot \bar{u}_\gamma(p'_2) \tilde{G}_{kl}^{(tc)}(p_2, p'_2 | A) u_\delta(p_2) \rangle_A . \end{aligned} \quad (5.12)$$

$[i, j, k, l \in \{1, \dots, N_c\}]$  are colour indices, while  $\alpha, \beta, \gamma, \delta \in \{1, 2\}$  are spin indices.]

The expectation value  $\langle O(A) \rangle_A$  of an arbitrary functional  $O(A)$  of the gluon field  $A^\mu$  is now defined as:

$$\langle O(A) \rangle_A \equiv \frac{1}{Z_{QCD}} \int [dA] O(A) \exp \left[ -\frac{i}{4} \int d^4x F_a^{\mu\nu} F_{a\mu\nu} \right] \det[i\gamma^\mu D_\mu - m] , \quad (5.13)$$

where  $Z_{QCD}$  is the partition function for fermion QCD:

$$\begin{aligned} Z_{QCD} & \equiv \int [dA][d\psi][d\psi^\dagger] \exp \left[ i \int d^4x L(\psi, \psi^\dagger, A) \right] \\ & = \int [dA] \exp \left[ -\frac{i}{4} \int d^4x F_a^{\mu\nu} F_{a\mu\nu} \right] \det[i\gamma^\mu D_\mu - m] . \end{aligned} \quad (5.14)$$

The determinant in Eqs. (5.13) and (5.14) comes from the integration over the fermion degrees of freedom.

If one uses the result (3.18) for  $\bar{u}_\alpha(p')\tilde{G}_{ij}^{(tc)}(p, p'|A)u_\beta(p)$ , one finally finds the following expression for the high-energy quark-quark elastic scattering amplitude in (fermion) QCD [7, 8, 9]:

$$\begin{aligned} \langle \psi_{i\alpha}(p'_1)\psi_{k\gamma}(p'_2)|T|\psi_{j\beta}(p_1)\psi_{l\delta}(p_2)\rangle &\simeq \\ &\simeq -\frac{i}{Z_W^2} \cdot \delta_{\alpha\beta}\delta_{\gamma\delta} \cdot 2s \int d^2\vec{z}_\perp e^{i\vec{q}_\perp \cdot \vec{z}_\perp} \langle [W_{p_1}(z_t) - \mathbf{1}]_{ij}[W_{p_2}(0) - \mathbf{1}]_{kl}\rangle_A. \end{aligned} \quad (5.15)$$

The notation is the same as for Eq. (5.10). In a perfectly analogous way, one can also derive the high-energy scattering amplitude for an elastic process involving two partons, which can be quarks, antiquarks or gluons. One simply has to insert in the matrix element, for each of the two partons involved, the corresponding quantity among (2.24), (2.30), (3.18), (3.19) and (4.11), which, as we have said before, describe the scattering amplitude of the parton in a given external gluon field. So, for example, the amplitude for the quark-gluon scattering

$$\psi_{j\beta}(p_1) + g_{a\lambda}(p_2) \rightarrow \psi_{i\alpha}(p'_1) + g_{a'\lambda'}(p'_2) \quad (5.16)$$

has the following expression in the high-energy limit [8]:

$$\begin{aligned} \langle \psi_{i\alpha}(p'_1)g_{a'\lambda'}(p'_2)|T|\psi_{j\beta}(p_1)g_{a\lambda}(p_2)\rangle &\simeq \\ &\simeq \frac{1}{Z_W Z_V} \langle \bar{u}_\alpha(p'_1)\tilde{G}_{ij}^{(tc)}(p_1, p'_1|A)u_\beta(p_1) \cdot \varepsilon_{(\lambda')}^{\mu'*}(p'_2)\tilde{D}_{\mu'\mu, a'a}^{(tc)}(p_2, p'_2|A)\varepsilon_{(\lambda)}^\mu(p_2)\rangle \\ &\simeq -\frac{i}{Z_W Z_V} \cdot \delta_{\alpha\beta}\delta_{\lambda'\lambda} \cdot 2s \int d^2\vec{z}_\perp e^{i\vec{q}_\perp \cdot \vec{z}_\perp} \langle [W_{p_1}(z_t) - \mathbf{1}]_{ij}[\mathcal{V}_{p_2}(0) - \mathbf{1}]_{a'a}\rangle_A. \end{aligned} \quad (5.17)$$

(The renormalization constant  $Z_V$  is defined and evaluated in the Appendix.) Let us observe at this point that, differently from their quark counterparts (which really come from the integration over the quark degrees of freedom in the functional integral), the gluon matrix element  $\varepsilon_{(\lambda')}^{\mu'*}(k')\tilde{D}_{\mu'\mu, a'a}^{(tc)}(k, k'|A)\varepsilon_{(\lambda)}^\mu(k)$  can only be defined as the proper functional of  $A^\mu$  which, when inserted in the functional average  $\langle \dots \rangle_A$  (together with another parton matrix element) reproduces the corresponding gluon-parton scattering amplitude in the high-energy limit. We have derived the expression (4.11) for the gluon matrix element by resumming all diagrams of the type reported in Fig. 1b in the eikonal limit. Therefore, e.g., Compton-like diagrams (and their perturbative corrections) are not included in Eq.

(5.17): indeed, one can easily convince oneself (see, for example, Ref. [4]) that these diagrams (which are present also in the Abelian case) are of order  $\mathcal{O}(s^0)$  in the high-energy limit  $s \rightarrow \infty$ ; while, for example, the  $\mathcal{O}(g^2)$  diagram described by Eq. (5.17) (i.e., the one obtained taking the first term in the perturbative expansion of the gluon matrix element and of the quark matrix element) is of order  $\mathcal{O}(s)$ . However, the expression (5.17), by its own construction, is not able to reproduce all those diagrams where the scattering gluon converts in a quark–antiquark pair during the diffusion process. Without resorting to large- $N_c$  approximations, where these diagrams are of course sub-leading corrections, a reason for not including these diagrams in the high-energy scattering amplitude can come if we first study “soft” high-energy parton–parton scattering in the “femto universe”, in the spirit of Refs. [7, 8]. In other words, we first consider the scattering of the partons over the finite time interval  $-t_0/2 \leq t \leq t_0/2$  ( $t = 0$  being the nominal collision time of the hadrons  $h_1 + h_2 \rightarrow h_1 + h_2$  in the c.m.s.) of length  $t_0 \approx 2$  fm:  $t_0/2$  is the time when, in an inelastic collision, the first produced hadrons appear. (The estimate  $t_0 \approx 2$  fm is discussed in Ref. [7].) As discussed in Refs. [7, 8], one can assume that over that time interval: a) parton annihilation and production processes can be neglected (i.e., the parton state of the hadrons does not change qualitatively in this time); b) partons travel in essence on straight lightlike world lines and they undergo “soft” elastic scattering. This was the strategy adopted in Ref. [7] in order to study parton–parton “soft” high-energy scattering in QCD. Of course, free quarks and gluons do not exist in (zero-temperature) QCD, but, assuming that  $t_0 \approx 2$  fm is nearly infinitely long on the scale of the “femto universe”, one can use the standard LSZ reduction formulae to relate the partonic  $S$ -matrix element to an integral over the four-point function of the quark/gluon fields. After having solved the problem of parton–parton scattering, one has to fold the partonic  $S$ -matrix with the hadronic wave functions of the appropriate resolution to get the hadronic  $S$ -matrix elements. We want to stress that the result (5.15) for quark–quark scattering at high energies was derived in Ref. [7] under the crucial assumption that only gluon modes up to a fixed frequency contribute in the functional integral  $\langle \dots \rangle_A$ . It was argued at length in Ref. [7] that this should be a valid approximation for the scattering of partons over the time interval  $(-t_0/2, t_0/2)$ , since, for these scattering processes, the relevant scale for the frequency of the exchanged quanta is given by  $a^{-1}$ , independent of  $s$ , where  $a$  is the correlation length of the (gauge-invariant) two-point function of the gluon field-strength tensor. In the hypothetical scattering amplitude for “real” quarks, all the splitting processes with long time scales will play an important role and therefore must be included.

## 6. Conclusions and outlook

In this paper we have derived the same results already found in Refs. [7, 8, 9] in a different and even more immediate way, i.e., by a direct resummation of perturbation theory in the limit of very high energy and small transferred momentum. The approximations used in the previous sections in order to derive the eikonal propagators (and, as a consequence, the parton-parton scattering amplitudes) in the above-mentioned limit are exactly the same adopted in Ref. [4] in order to derive the high-energy asymptotic expressions for the elastic parton-parton scattering amplitudes at a given order in perturbation theory, using standard Feynman-diagrams techniques. (These approximations essentially consist in neglecting the recoil of the scattering partons in the diffusion process.) Therefore, in this sense, we can claim that we have given a proof that the nonperturbative expressions (5.10), (5.15), etc., for the high-energy elastic scattering amplitude are a resummation of the corresponding perturbative results found in Ref. [4]. (In a previous paper [13], we gave a direct proof that this is true up to the fourth order in the expansion in the coupling constant.)

We remind the reader that the  $s$  dependence of the scattering amplitude is not all contained in the kinematical factor  $2s$  in front of the integral in Eqs. (5.10), (5.15), etc. . In fact, as was first pointed out by Verlinde and Verlinde in [19], it is a singular limit to take the Wilson lines in (5.10), (5.15), etc., exactly lightlike. It turns out that a proper regularization of these “infrared” singularities (so called because they essentially come from the limit  $m \rightarrow 0$ ,  $m$  being the quark mass) gives rise to a  $\log s$  dependence of the amplitude, as obtained by ordinary perturbation theory [4, 5] and as confirmed by the experiments on hadron-hadron scattering processes. In practice, the regularization procedure consists in letting each Wilson line have a small timelike component (so that they coincide with the classical trajectories for quarks with a finite mass  $m$ ) and letting them end after some finite proper time  $\pm T$ , i.e., after some time  $|t| \sim T\sqrt{s}/2m$  in the c.m.s. (so that, when one takes the limit  $s \rightarrow \infty$ , the trajectories of the Wilson lines will again become infinitely long and lightlike). We refer the reader to Refs. [19] and [13, 14, 15] for a detailed discussion about this point.

We want also to stress that the expressions (5.10), (5.15), etc., for the scattering amplitudes are not limited by our approximations to be “*quenched*”: indeed, in the functional average  $\langle \dots \rangle_A$  also the determinant of the quark matrix is included and it gives rise to Feynman diagrams with dynamical quark loops in the perturbative expansion. So, for

example, the so-called “*tower diagrams*” for quark–quark elastic scattering, which give amplitudes as large as  $s(\log s)^n$ , where  $n$  is the number of quark loops joined together vertically, with each loop having four vertices (see, e.g., Ref. [4] and references therein), are expected to be reproduced in the perturbative expansion of Eqs. (5.10), (5.15), etc., provided you integrate from  $t = -\infty$  to  $t = +\infty$ , i.e., you take infinitely long Wilson lines, since the time associated with the hard fluctuations in these eikonal processes is very large. [If we truncate the Wilson lines from  $t = -t_0$  to  $t = +t_0$  (in the c.m.s. of the reaction), with  $t_0$  fixed as  $s \rightarrow \infty$ , we can lose in our description all those processes in which there is production of hard partons, whose fluctuation time is larger than  $t_0$  (in the asymptotic limit  $s \rightarrow \infty$ ). These processes must then be included in the wave functions of the fermions at  $t = \pm t_0$ .]

As we have said before, free asymptotic states of quarks and gluons do not exist in (zero-temperature) QCD. Therefore, in this case, the correct procedure should be the one suggested in Refs. [7, 8], where Eqs. (5.10), (5.15), etc., are considered to be the partonic  $S$ -matrix elements in the “femto universe” (with a fixed frequency cutoff for the gluon modes in the functional average): they must be folded with the hadronic wave functions of the appropriate resolution to get the hadronic  $S$ -matrix elements.

Viceversa, in QED one can have the free-electron states as IN and OUT asymptotic states: there is no complication due to confinement and a free-electron state is a well-defined asymptotic state of the theory. Therefore, the eikonal formula (5.15) in QED, with infinitely long Wilson lines, can be really considered as the asymptotic expression for the fermion–fermion elastic scattering amplitude in the high-energy limit and it should reproduce all perturbative results evaluated by Cheng and Wu in Ref. [4]. We want to stress that the eikonal formula (5.15) is not identical to what, in the literature, is usually called the “eikonal amplitude” of the high-energy scattering in QED (see Refs. [16, 17, 18]): this last was obtained in the so-called “*quenched*” approximation, where vacuum polarization effects, arising from the presence of dynamical fermion loops, are neglected. It was proved in Ref. [9] (see also Refs. [21, 22]) that, when evaluating Eq. (5.15) in the *quenched* approximation, one correctly reproduces the eikonal result of Refs. [16, 17, 18]. However, as we have said before, the amplitude (5.15) is not limited, in general, to be *quenched* and so it is expected to be more correct and to contain more information than the eikonal result of Refs. [16, 17, 18]).

Once we have found the nonperturbative expressions (5.10), (5.15), etc., for the high-energy scattering amplitudes, the natural question which arises is: How can we evaluate



them directly? The answer to this question is highly nontrivial and it is also strictly connected with the renormalization properties of Wilson–line operators [20, 21]. Some nonperturbative approaches for the calculation of (5.15) were proposed in Refs. [22] and [23]. In particular, in Refs. [23, 24] a nonperturbative numerical estimate of high–energy hadron–hadron scattering amplitudes was obtained, using Eq. (5.15) as a basic ingredient, in the framework of the so–called “stochastic vacuum model”. For an alternative approach to the problem, we refer the reader to Refs. [13, 14, 15], where interesting analytic properties of the high–energy scattering amplitude were derived, going from Minkowskian to Euclidean space–time, so opening the possibility of studying the high–energy scattering amplitude using the Euclidean formulation of the theory, e.g., on the lattice. (See also Ref. [25], where a similar analytic continuation from Minkowskian to Euclidean theory was proposed to study the small- $x_{Bj}$  behaviour of the structure functions of deep inelastic lepton–nucleon scattering.) The analytic continuation proposed in Refs. [13, 14, 15] has been recently adopted in Ref. [26], in order to study the high–energy scattering in  $\mathcal{N} = 4$  supersymmetric  $SU(N_c)$  gauge theories (in the strong coupling, large- $N_c$  limit) using the AdS/CFT correspondence, and also in Ref. [27], in order to investigate instanton–induced effects in QCD high–energy scattering. In our opinion, a considerable progress could be achieved by a direct investigation of these problems on the lattice in the near future.

## Acknowledgements

I would like to thank Prof. Otto Nachtmann for his useful suggestions and comments and also for having encouraged me to write this paper.

## Appendix: The residue at the pole of the unrenormalized eikonal propagators

### A.1 The scalar propagator

Let us consider now the unrenormalized full scalar propagator (not truncated!) in the vicinity of the pole (i.e., for  $p^2 \rightarrow m^2$ ). We start from the scalar propagator  $S_{ij}(x, y|A)$  in an external gluon field  $A^\mu$  and then we take the functional average over the gluon field in order to get the full propagator (2.2):

$$\langle S_{ij}(x, y|A) \rangle_A = \langle T[\phi_i(x)\phi_j^\dagger(y)] \rangle = S_{ij}(x - y) . \quad (\text{A.1})$$

Therefore, using the definition (2.9), we find:

$$\langle \tilde{S}_{ij}(p, p'|A) \rangle_A = (2\pi)^4 \delta^{(4)}(p' - p) \tilde{S}_{ij}(p) , \quad (\text{A.2})$$

where  $\tilde{S}_{ij}(p)$  is the unrenormalized propagator in the momentum space, defined by Eq. (2.4). We shall evaluate the quantity (A.2) starting from the perturbative expansion of  $\tilde{S}_{ij}^{(tc)}(p, p'|A)$  in the eikonal approximation, that we have computed in Sect. 2. At the 0-th order we find the following expression for  $\langle \tilde{S}_{ij}(p, p'|A) \rangle_A$ :

$$[\langle \tilde{S}_{ij}(p, p'|A) \rangle_A]_{(0)} = (2\pi)^4 \delta^{(4)}(p' - p) \frac{i}{p^2 - m^2 + i\varepsilon} . \quad (\text{A.3})$$

At the  $n$ -th order ( $n \geq 1$ ) we find, using the corresponding expression for  $[\tilde{S}_{ij}^{(tc)}(p, p'|A)]_{(n)}$  that we have derived in Sect. 2 [see Eq. (2.20)]:

$$\begin{aligned} [\langle \tilde{S}_{ij}(p, p'|A) \rangle_A]_{(n)} &\simeq \\ &\simeq \frac{i}{p'^2 - m^2 + i\varepsilon} \int d^2 \vec{b}_{n\perp} \int db_{n-} \int db_{n+} \dots \int db_{1+} e^{iqb_n} \\ &\times \frac{1}{(2E)^{n-1}} \theta(b_{n+} - b_{(n-1)+}) \dots \theta(b_{2+} - b_{1+}) \\ &\times \langle \{ [-igp^{\mu_n} A_{\mu_n}(b_n)] \dots [-igp^{\mu_1} A_{\mu_1}(b_1)] \}_{ij} \rangle_A |_{b_{i-}=b_{n-} ; \vec{b}_{i\perp}=\vec{b}_{n\perp}} \\ &\times \frac{i}{p^2 - m^2 + i\varepsilon} , \end{aligned} \quad (\text{A.4})$$

where  $q = p' - p$ . By virtue of the invariance under the Poincarè group (in particular, under space-time translations):

$$\langle A_{\mu_n}(b_n) A_{\mu_{n-1}}(b_{n-1}) \dots A_{\mu_1}(b_1) \rangle_A = \langle A_{\mu_n}(0) A_{\mu_{n-1}}(b_{n-1} - b_n) \dots A_{\mu_1}(b_1 - b_n) \rangle_A . \quad (\text{A.5})$$

Therefore, making the change of variables

$$c_i = b_i - b_n, \quad \forall i = 1, \dots, n-1 ; \quad c_n = b_n , \quad (\text{A.6})$$

the integration over  $c_n$  gives  $2(2\pi)^4 \delta^{(4)}(q)$  and we can write Eq. (A.4) as follows:

$$\begin{aligned} & [\langle \tilde{S}_{ij}(p, p' | A) \rangle_A]_{(n)} \simeq \\ & \simeq 2(2\pi)^4 \delta^{(4)}(q) \frac{i}{p'^2 - m^2 + i\varepsilon} \int dc_{(n-1)+} \dots \int dc_{1+} \\ & \times \frac{1}{(2E)^{n-1}} \theta(-c_{(n-1)+}) \theta(c_{(n-1)+} - c_{(n-2)+}) \dots \theta(c_{2+} - c_{1+}) \\ & \times \langle \{ [-igp^{\mu_n} A_{\mu_n}(0)] [-igp^{\mu_{n-1}} A_{\mu_{n-1}}(c_{n-1})] \dots [-igp^{\mu_1} A_{\mu_1}(c_1)] \}_{ij} \rangle_A |_{c_{i-}=0 ; \vec{c}_{i\perp}=\vec{0}_\perp} \\ & \times \frac{i}{p^2 - m^2 + i\varepsilon} . \end{aligned} \quad (\text{A.7})$$

In the vicinity of the pole, i.e., for  $p^2 \rightarrow m^2$ , one has that:  $p'^2 - m^2 \simeq 2pq + q^2$ . Moreover, in the eikonal limit,  $p_+ \simeq 2E$ ,  $p_- \simeq 0$  and  $\vec{p}_\perp \simeq \vec{0}_\perp$ , so that:

$$\frac{i}{p'^2 - m^2 + i\varepsilon} \simeq \frac{i}{2E} \frac{1}{q_- + i\varepsilon} = \frac{i}{2E} \mathcal{P} \frac{1}{q_-} + \frac{\pi}{2E} \delta(q_-) , \quad (\text{A.8})$$

where “ $\mathcal{P}$ ” stands for “principal-part value”. While  $\delta^{(4)}(q) \mathcal{P}(1/q_-) = 0$ , we can write, formally:

$$2\delta^{(4)}(q) \frac{\pi}{2E} \delta(q_-) = \delta^{(4)}(q) \frac{1}{2E} \int_{-\infty}^{+\infty} dc_{n+} e^{ic_{n+}q_-} = \delta^{(4)}(q) \frac{1}{2E} \int_{-\infty}^{+\infty} dc_{n+} . \quad (\text{A.9})$$

(The origin of this infrared singularity is discussed in Sect. 6 and can be “cured” by a proper regularization of the Wilson lines, as suggested in Refs. [19] and [13, 14, 15].) Substituting in Eq. (A.7) and changing again the integration variables from  $c_i$  to  $b_i =$

$c_i + c_n$ ,  $\forall i = 1, \dots, n-1$  and  $b_n = c_n$ , we obtain:

$$\begin{aligned}
& [\langle \tilde{S}_{ij}(p, p' | A) \rangle_A]_{(n)} \simeq \\
& \simeq (2\pi)^4 \delta^{(4)}(q) \frac{i}{p^2 - m^2 + i\varepsilon} \int db_{n+} \dots \int db_{1+} \\
& \times \frac{1}{(2E)^n} \theta(b_{n+} - b_{(n-1)+}) \dots \theta(b_{2+} - b_{1+}) \\
& \times \langle \{ [-igp^{\mu_n} A_{\mu_n}(b_n)] \dots [-igp^{\mu_1} A_{\mu_1}(b_1)] \}_{ij} \rangle_A |_{b_{i-}=b_{n-}} ; \vec{b}_{i\perp} = \vec{b}_{n\perp} . \quad (\text{A.10})
\end{aligned}$$

(We have again made use of the invariance under space-time translations in  $\langle \dots \rangle_A$ .) We can parametrize the coordinates  $b_i$  ( $i = 1, \dots, n$ ) as in Sect. 2, i.e., in the form  $b_i = b + p\tau_i$  ( $i = 1, \dots, n$ ), with a fixed  $b$ . In fact, using  $p_+ \simeq 2E$ ,  $p_- \simeq 0$  and  $\vec{p}_\perp \simeq \vec{0}_\perp$ , one has that  $b_{i-} = 0$ ,  $\vec{b}_{i\perp} = \vec{0}_\perp$  and  $db_{i+} = 2Ed\tau_i$ . Eq. (A.10) becomes:

$$\begin{aligned}
& [\langle \tilde{S}_{ij}(p, p' | A) \rangle_A]_{(n)} \simeq \\
& \simeq (2\pi)^4 \delta^{(4)}(q) \frac{i}{p^2 - m^2 + i\varepsilon} \int d\tau_1 \dots \int d\tau_n \theta(\tau_n - \tau_{n-1}) \dots \theta(\tau_2 - \tau_1) \\
& \times \langle \{ [-igp^{\mu_n} A_{\mu_n}(b + p\tau_n)] \dots [-igp^{\mu_1} A_{\mu_1}(b + p\tau_1)] \}_{ij} \rangle_A . \quad (\text{A.11})
\end{aligned}$$

In conclusion, summing all perturbative orders [Eqs. (A.3) and (A.11)], we find the following expression for  $\langle \tilde{S}_{ij}(p, p' | A) \rangle_A$ :

$$\begin{aligned}
& \langle \tilde{S}_{ij}(p, p' | A) \rangle_A \simeq \\
& \simeq (2\pi)^4 \delta^{(4)}(q) \frac{i}{p^2 - m^2 + i\varepsilon} \cdot \langle [W_p(b)]_{ij} \rangle_A \\
& = (2\pi)^4 \delta^{(4)}(q) \frac{i}{p^2 - m^2 + i\varepsilon} \cdot \frac{\delta_{ij}}{N_c} \langle \text{Tr}[W_p(b)] \rangle_A , \quad (\text{A.12})
\end{aligned}$$

where  $W_p(b)$  has been defined in Eq. (2.25). (In the last passage we have used the fact that the vacuum is gauge-invariant.) By virtue of Eq. (A.2), we derive the following expression for  $\tilde{S}_{ij}(p)$  in the vicinity of the pole:

$$\tilde{S}_{ij}(p) \underset{p^2 \rightarrow m^2}{\simeq} \frac{i}{p^2 - m^2 + i\varepsilon} \cdot \frac{\delta_{ij}}{N_c} \langle \text{Tr}[W_p(b)] \rangle_A . \quad (\text{A.13})$$

Moreover, from the definition of the “physical” mass  $m$  and of the residue  $Z_W$  at the pole for  $p^2 \rightarrow m^2$  of the unrenormalized quark propagator, we must have [see Eq. (2.3)]:

$$\tilde{S}_{ij}(p) \underset{p^2 \rightarrow m^2}{\simeq} \frac{iZ_W \delta_{ij}}{p^2 - m^2 + i\varepsilon} . \quad (\text{A.14})$$

From the comparison of Eqs. (A.13) and (A.14), we derive the following expression for the residue  $Z_W$  at the pole:

$$Z_W = \frac{1}{N_c} \langle \text{Tr}[W_p(b)] \rangle_A = \frac{1}{N_c} \langle \text{Tr}[W_p(0)] \rangle_A , \quad (\text{A.15})$$

where we have again made use of the invariance under space-time translations.

## A.2 The fermion propagator

Let us consider now the unrenormalized full fermion propagator (not truncated!) in the vicinity of the pole (i.e., for  $p^2 \rightarrow m^2$ ). We start from the fermion propagator  $G_{ij}(x, y|A)$  in an external gluon field  $A^\mu$  and then we take the functional average over the gluon field in order to get the full propagator (3.2):

$$\langle G_{ij}(x, y|A) \rangle_A = \langle T[\psi_i(x) \bar{\psi}_j(y)] \rangle = G_{ij}(x - y) . \quad (\text{A.16})$$

Going to the momentum representation [see Eq. (2.9)], we obtain:

$$\langle \tilde{G}_{ij}(p, p'|A) \rangle_A = (2\pi)^4 \delta^{(4)}(p' - p) \tilde{G}_{ij}(p) , \quad (\text{A.17})$$

where  $\tilde{G}_{ij}(p)$  is the unrenormalized propagator in the momentum space, defined by Eq. (3.4). More precisely, we shall evaluate the following quantity:

$$\bar{u}_\alpha(p') \langle \tilde{G}_{ij}(p, p'|A) \rangle_A u_\beta(p) = (2\pi)^4 \delta^{(4)}(p' - p) \bar{u}_\beta(p) \tilde{G}_{ij}(p) u_\alpha(p) , \quad (\text{A.18})$$

starting from the perturbative expansion of  $\tilde{G}_{ij}^{(tc)}(p, p'|A)$  in the eikonal approximation, that we have computed in Sect. 3. At the 0-th order we find the following expression for the quantity (A.18):

$$\begin{aligned} & [\bar{u}_\alpha(p') \langle \tilde{G}_{ij}(p, p'|A) \rangle_A u_\beta(p)]_{(0)} = \\ & = (2\pi)^4 \delta^{(4)}(p' - p) \bar{u}_\alpha(p') \frac{i}{\hat{p} - m + i\varepsilon} u_\beta(p) \\ & = (2\pi)^4 \delta^{(4)}(p' - p) 4m^2 \delta_{\alpha\beta} \frac{i}{p^2 - m^2 + i\varepsilon} . \end{aligned} \quad (\text{A.19})$$

At the  $n$ -th order ( $n \geq 1$ ) we find, using the corresponding expression for  $[\tilde{G}_{ij}^{(tc)}(p, p'|A)]_{(n)}$  that we have derived in Sect. 3 [see Eq. (3.12)]:

$$\begin{aligned} & [\bar{u}_\alpha(p') \langle \tilde{G}_{ij}(p, p'|A) \rangle_A u_\beta(p)]_{(n)} = \\ &= \frac{i}{p^2 - m^2 + i\varepsilon} \int \frac{d^4 q_1}{(2\pi)^4} \cdots \int \frac{d^4 q_n}{(2\pi)^4} (2\pi)^4 \delta^{(4)}(q - q_1 - \dots - q_n) Q_{\alpha\beta, ij}^{(ferm)}(q_1, \dots, q_n) \\ & \times \frac{i}{(p + q_1 + \dots + q_{n-1})^2 - m^2 + i\varepsilon} \cdots \frac{i}{(p + q_1)^2 - m^2 + i\varepsilon} \cdot \frac{i}{p^2 - m^2 + i\varepsilon}, \end{aligned} \quad (\text{A.20})$$

where  $q = p' - p$  and  $Q_{\alpha\beta, ij}^{(ferm)}(q_1, \dots, q_n)$  is given by:

$$\begin{aligned} & Q_{\alpha\beta, ij}^{(ferm)}(q_1, \dots, q_n) \equiv \\ & \equiv \langle \bar{u}_\alpha(p') (\hat{p}' + m) \{ [-ig\gamma^{\mu_n} \tilde{A}_{\mu_n}(q_n) + i\delta m (2\pi)^4 \delta^{(4)}(q_n) \cdot \mathbf{1}] (\hat{p} + \hat{q}_1 + \dots + \hat{q}_{n-1} + m) \\ & \dots (\hat{p} + \hat{q}_1 + m) [-ig\gamma^{\mu_1} \tilde{A}_{\mu_1}(q_1) + i\delta m (2\pi)^4 \delta^{(4)}(q_1) \cdot \mathbf{1}] \}_{ij} (\hat{p} + m) u_\beta(p) \rangle_A \\ & = 4m^2 \langle N_{\alpha\beta, ij}^{(ferm)}(q_1, \dots, q_n) \rangle_A, \end{aligned} \quad (\text{A.21})$$

where  $N_{\alpha\beta, ij}^{(ferm)}(q_1, \dots, q_n)$  has been defined in Eq. (3.13). By using the result (3.16), we find that the expression (A.20) simplifies as follows:

$$\begin{aligned} & [\bar{u}_\alpha(p') \langle \tilde{G}_{ij}(p, p'|A) \rangle_A u_\beta(p)]_{(n)} \simeq 4m^2 \delta_{\alpha\beta} \cdot [\langle [\tilde{S}_{ij}(p, p'|A)]_A \rangle_{(n)}] \\ & \simeq 4m^2 \delta_{\alpha\beta} \cdot (2\pi)^4 \delta^{(4)}(q) \frac{i}{p^2 - m^2 + i\varepsilon} \int d\tau_1 \dots \int d\tau_n \theta(\tau_n - \tau_{n-1}) \dots \theta(\tau_2 - \tau_1) \\ & \times \langle \{ [-igp^{\mu_n} A_{\mu_n}(b + p\tau_n)] \dots [-igp^{\mu_1} A_{\mu_1}(b + p\tau_1)] \}_{ij} \rangle_A. \end{aligned} \quad (\text{A.22})$$

In conclusion, summing all perturbative orders [Eqs. (A.19) and (A.22)], we obtain the following expression for the quantity (A.18), evaluated in the vicinity of the pole:

$$\begin{aligned} & \bar{u}_\alpha(p') \langle \tilde{G}_{ij}(p, p'|A) \rangle_A u_\beta(p) \simeq 4m^2 \delta_{\alpha\beta} \cdot \langle \tilde{S}_{ij}(p, p'|A) \rangle_A \\ & \simeq 4m^2 \delta_{\alpha\beta} \cdot (2\pi)^4 \delta^{(4)}(q) \frac{i}{p^2 - m^2 + i\varepsilon} \cdot \frac{\delta_{ij}}{N_c} \langle \text{Tr}[W_p(b)] \rangle_A, \end{aligned} \quad (\text{A.23})$$

where  $W_p(b)$  has been defined in Eq. (2.25). (In the last passage we have used the fact that the vacuum is gauge-invariant.) By virtue of Eq. (A.18), we derive the following expression for  $\bar{u}_\alpha(p) \tilde{G}_{ij}(p) u_\beta(p)$  in the vicinity of the pole:

$$\bar{u}_\alpha(p) \tilde{G}_{ij}(p) u_\beta(p) \underset{p^2 \rightarrow m^2}{\simeq} 4m^2 \delta_{\alpha\beta} \cdot \frac{i}{p^2 - m^2 + i\varepsilon} \cdot \frac{\delta_{ij}}{N_c} \langle \text{Tr}[W_p(b)] \rangle_A. \quad (\text{A.24})$$

Moreover, from the definition of the “physical” mass  $m$  and of the residue  $Z_W$  at the pole for  $p^2 \rightarrow m^2$  of the unrenormalized quark propagator, we must have [see Eq. (3.3)]:

$$\tilde{G}_{ij}(p) \underset{p^2 \rightarrow m^2}{\simeq} \frac{iZ_W \delta_{ij}}{\hat{p} - m + i\varepsilon} = \frac{iZ_W \delta_{ij} (\hat{p} + m)}{p^2 - m^2 + i\varepsilon} , \quad (\text{A.25})$$

and, therefore:

$$\bar{u}_\alpha(p) \tilde{G}_{ij}(p) u_\beta(p) \underset{p^2 \rightarrow m^2}{\simeq} 4m^2 \delta_{\alpha\beta} \cdot \frac{iZ_W \delta_{ij}}{p^2 - m^2 + i\varepsilon} . \quad (\text{A.26})$$

From the comparison of Eqs. (A.24) and (A.26), we derive the following expression for the residue  $Z_W$  at the pole:

$$Z_W = \frac{1}{N_c} \langle \text{Tr}[W_p(b)] \rangle_A = \frac{1}{N_c} \langle \text{Tr}[W_p(0)] \rangle_A , \quad (\text{A.27})$$

where we have made use of the invariance under space–time translations. This expression is formally identical to the one we have obtained for the scalar case: however, we must remember that now  $\langle \dots \rangle_A$  has to be intended as the functional average over the gluon field in the theory with fermions. The same result (A.27) was also derived in Ref. [7] using a different method.

### A.3 The gluon propagator

Let us consider, finally, the unrenormalized full gluon propagator (not truncated!) in the vicinity of the pole (i.e., for  $k^2 \rightarrow 0$ ). We start from the “gluon propagator”  $D_{\mu'\mu}^{a'a}(x, y|A)$  in an external gluon field  $A_b^\nu$  and then we take the functional average over the gluon field  $A_b^\nu$  in order to get the full propagator:

$$\langle D_{\mu'\mu}^{a'a}(x, y|A) \rangle_A = \langle T[A_{\mu'}^{a'}(x) A_\mu^a(y)] \rangle = D_{\mu'\mu}^{a'a}(x - y) . \quad (\text{A.28})$$

Going to the momentum representation [see Eq. (2.9)], we obtain:

$$\langle \tilde{D}_{\mu'\mu}^{a'a}(k, k'|A) \rangle_A = (2\pi)^4 \delta^{(4)}(k' - k) \tilde{D}_{\mu'\mu}^{a'a}(k) , \quad (\text{A.29})$$

where  $\tilde{D}_{\mu'\mu}^{a'a}(k)$  is the full propagator in the momentum space:

$$\tilde{D}_{\mu'\mu}^{a'a}(k) \equiv \int d^4z \, e^{ikz} D_{\mu'\mu}^{a'a}(z) . \quad (\text{A.30})$$

More precisely, we shall evaluate the following quantity:

$$\varepsilon_{(\lambda')}^{\mu'*}(k') \langle \tilde{D}_{\mu'\mu}^{a'a}(k, k'|A) \rangle_A \varepsilon_{(\lambda)}^\mu(k) = (2\pi)^4 \delta^{(4)}(k' - k) \varepsilon_{(\lambda')}^{\mu'*}(k) \tilde{D}_{\mu'\mu}^{a'a}(k) \varepsilon_{(\lambda)}^\mu(k) , \quad (\text{A.31})$$

starting from the perturbative expansion of  $\tilde{D}_{\mu'\mu, a'a}^{(tc)}(k, k'|A)$  in the eikonal approximation, that we have computed in Sect. 4. At the 0-th order we find the following expression for the quantity (A.31):

$$\left[ \varepsilon_{(\lambda')}^{\mu'*}(k') \langle \tilde{D}_{\mu'\mu}^{a'a}(k, k'|A) \rangle_A \varepsilon_{(\lambda)}^\mu(k) \right]_{(0)} = (2\pi)^4 \delta^{(4)}(k' - k) \delta_{\lambda'\lambda} \delta_{a'a} \frac{i}{k^2 + i\varepsilon} . \quad (\text{A.32})$$

At the  $n$ -th order ( $n \geq 1$ ) we find, using the corresponding expression for the “truncated–connected propagator”  $[\tilde{D}_{\mu'\mu}^{a'a}(k, k'|A)]_{(n)}$  that we have derived in Sect. 4 [see Eqs. (4.5) – (4.7)]:

$$\begin{aligned} & \left[ \varepsilon_{(\lambda')}^{\mu'*}(k') \langle \tilde{D}_{\mu'\mu}^{a'a}(k, k'|A) \rangle_A \varepsilon_{(\lambda)}^\mu(k) \right]_{(n)} \simeq \\ & \simeq \delta_{\lambda'\lambda} \frac{i}{k'^2 + i\varepsilon} \int \frac{d^4 q_1}{(2\pi)^4} \cdots \int \frac{d^4 q_n}{(2\pi)^4} (2\pi)^4 \delta^{(4)}(q - q_1 - \cdots - q_n) \\ & \times \langle \{ [-ig2k^{\mu_n} \tilde{\mathcal{A}}_{\mu_n}(q_n)] \cdots [-ig2k^{\mu_1} \tilde{\mathcal{A}}_{\mu_1}(q_1)] \}_{a'a} \rangle_A \\ & \times \frac{i}{(k + q_1 + \cdots + q_{n-1})^2 + i\varepsilon} \cdots \frac{i}{(k + q_1)^2 + i\varepsilon} \cdot \frac{i}{k^2 + i\varepsilon} , \end{aligned} \quad (\text{A.33})$$

where  $q \equiv k' - k$  is the transferred momentum. This expression is perfectly analogous to the corresponding expression obtained in Sect. A.1 for the scalar case. We can proceed as in that case to simplify Eq. (A.33) as follows:

$$\begin{aligned} & \left[ \varepsilon_{(\lambda')}^{\mu'*}(k') \langle \tilde{D}_{\mu'\mu}^{a'a}(k, k'|A) \rangle_A \varepsilon_{(\lambda)}^\mu(k) \right]_{(n)} \simeq \\ & \simeq (2\pi)^4 \delta^{(4)}(q) \delta_{\lambda'\lambda} \frac{i}{k^2 + i\varepsilon} \int d\tau_1 \cdots \int d\tau_n \theta(\tau_n - \tau_{n-1}) \cdots \theta(\tau_2 - \tau_1) \\ & \times \langle \{ [-igk^{\mu_n} \mathcal{A}_{\mu_n}(b + k\tau_n)] \cdots [-igk^{\mu_1} \mathcal{A}_{\mu_1}(b + k\tau_1)] \}_{a'a} \rangle_A . \end{aligned} \quad (\text{A.34})$$

In conclusion, summing all perturbative orders [Eqs. (A.32) and (A.34)], we find the following expression for the quantity (A.31), evaluated in the vicinity of the pole:

$$\varepsilon_{(\lambda')}^{\mu'*}(k') \langle \tilde{D}_{\mu'\mu}^{a'a}(k, k'|A) \rangle_A \varepsilon_{(\lambda)}^\mu(k) \simeq (2\pi)^4 \delta^{(4)}(q) \delta_{\lambda'\lambda} \frac{i}{k^2 + i\varepsilon} \cdot \frac{\delta_{a'a}}{N_c^2 - 1} \langle \text{Tr}[\mathcal{V}_k(b)] \rangle_A , \quad (\text{A.35})$$



where  $\mathcal{V}_k(b)$  has been defined in Eq. (4.12) and we have made use of the gauge-invariance of the vacuum (i.e., of the functional average  $\langle \dots \rangle_A$ ). By virtue of Eq. (A.31), we derive the following expression for  $\varepsilon_{(\lambda')}^{\mu'*}(k) \tilde{D}_{\mu'\mu}^{a'a}(k) \varepsilon_{(\lambda)}^\mu(k)$  in the vicinity of the pole:

$$\varepsilon_{(\lambda')}^{\mu'*}(k) \tilde{D}_{\mu'\mu}^{a'a}(k) \varepsilon_{(\lambda)}^\mu(k) \underset{k^2 \rightarrow 0}{\simeq} \delta_{\lambda'\lambda} \frac{i}{k^2 + i\varepsilon} \cdot \frac{\delta_{a'a}}{N_c^2 - 1} \langle \text{Tr}[\mathcal{V}_k(b)] \rangle_A . \quad (\text{A.36})$$

Moreover, from the definition of the residue  $Z_\mathcal{V}$  at the pole for  $k^2 \rightarrow 0$  of the unrenormalized full gluon propagator, we must have that:

$$\tilde{D}_{\mu'\mu}^{a'a}(k) \underset{k^2 \rightarrow 0}{\simeq} \frac{-iZ_\mathcal{V} g_{\mu'\mu} \delta_{a'a}}{k^2 + i\varepsilon} , \quad (\text{A.37})$$

and, therefore:

$$\varepsilon_{(\lambda')}^{\mu'*}(k) \tilde{D}_{\mu'\mu}^{a'a}(k) \varepsilon_{(\lambda)}^\mu(k) \underset{k^2 \rightarrow 0}{\simeq} \delta_{\lambda'\lambda} \delta_{a'a} \frac{iZ_\mathcal{V}}{k^2 + i\varepsilon} . \quad (\text{A.38})$$

From the comparison of Eqs. (A.36) and (A.38), we derive the following expression for the residue  $Z_\mathcal{V}$  at the pole:

$$Z_\mathcal{V} = \frac{1}{N_c^2 - 1} \langle \text{Tr}[\mathcal{V}_k(b)] \rangle_A = \frac{1}{N_c^2 - 1} \langle \text{Tr}[\mathcal{V}_k(0)] \rangle_A , \quad (\text{A.39})$$

where we have made use of the invariance under space-time translations.

## References

- [1] T. Regge, *Nuovo Cimento* **14**, 951 (1959); **18**, 947 (1960).
- [2] N. Byers and C.N. Yang, *Phys. Rev.* **142**, 976 (1966).
- [3] F.E. Low, *Phys. Rev. D* **12**, 163 (1975);  
S. Nussinov, *Phys. Rev. Lett.* **34**, 1286 (1975).
- [4] H. Cheng and T.T. Wu, *Expanding Protons: Scattering at High Energies* (MIT Press, Cambridge, Massachusetts, 1987).
- [5] L.N. Lipatov, in *Review in Perturbative QCD*, edited by A.H. Mueller (World Scientific, Singapore, 1989), and references therein.
- [6] P.V. Landshoff and O. Nachtmann, *Z. Phys. C* **35**, 405 (1987).
- [7] O. Nachtmann, *Ann. Phys.* **209**, 436 (1991).
- [8] O. Nachtmann, in *Perturbative and Nonperturbative aspects of Quantum Field Theory*, edited by H. Latal and W. Schweiger (Springer–Verlag, Berlin, Heidelberg, 1997).
- [9] E. Meggiolaro, *Phys. Rev. D* **53**, 3835 (1996).
- [10] E.S. Fradkin, *Proceedings of the International Winter School on Theoretical Physics at JINR* (Dubna, 1964); *Acta Phys. Hung.* XIX, 175 (1964).
- [11] H. Lehmann, K. Symanzik and W. Zimmermann, *Nuovo Cimento* **1**, 205 (1955).
- [12] M. Fabbrichesi, R. Pettorino, G. Veneziano and G.A. Vilkovisky, *Nucl. Phys. B* **419**, 147 (1994).
- [13] E. Meggiolaro, *Z. Phys. C* **76**, 523 (1997).
- [14] E. Meggiolaro, *Eur. Phys. J. C* **4**, 101 (1998).
- [15] E. Meggiolaro, *Nucl. Phys. B (Proc. Suppl.)* **64**, 191 (1998).
- [16] H. Cheng and T.T. Wu, *Phys. Rev. Lett.* **22**, 666 (1969).
- [17] H. Abarbanel and C. Itzykson, *Phys. Rev. Lett.* **23**, 53 (1969).

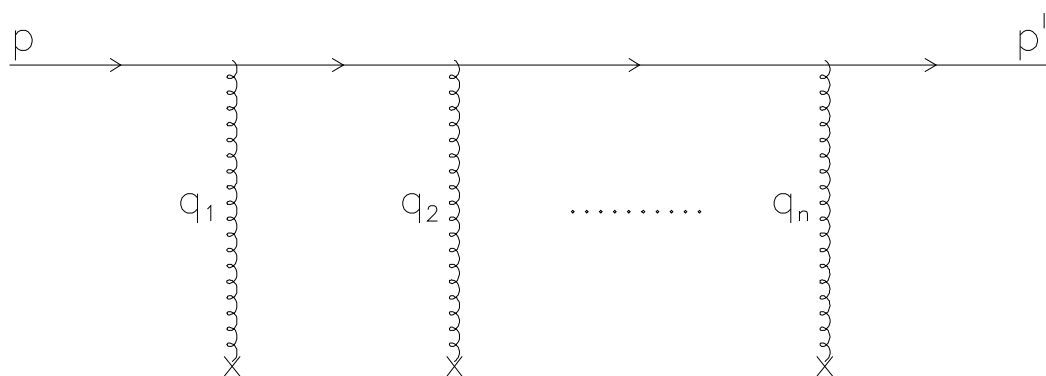
- [18] R. Jackiw, D. Kabat and M. Ortiz, Phys. Lett. B **277**, 148 (1992).
- [19] H. Verlinde and E. Verlinde, Princeton University, report No. PUPT-1319 (revised 1993); hep-th/9302104.
- [20] I.Ya. Aref'eva, Phys. Lett. **93B**, 347 (1980).
- [21] G.P. Korchemsky, Phys. Lett. B **325**, 459 (1994);  
I.A. Korchemskaya and G.P. Korchemsky, Nucl. Phys. B **437**, 127 (1995).
- [22] I.Ya. Aref'eva, Phys. Lett. B **325**, 171 (1994); **328**, 411 (1994).
- [23] H.G. Dosch, E. Ferreira and A. Krämer, Phys. Rev. D **50**, 1992 (1994).
- [24] E.R. Berger and O. Nachtmann, Eur. Phys. J. C **7**, 459 (1999).
- [25] A. Hebecker, E. Meggiolaro and O. Nachtmann, Nucl. Phys. B **571**, 26 (2000).
- [26] R.A. Janik and R. Peschanski, Nucl. Phys. B **565**, 193 (2000);  
R.A. Janik and R. Peschanski, hep-th/0003059.
- [27] E. Shuryak and I. Zahed, hep-ph/0005152.

## FIGURE CAPTIONS

**Fig. 1.** a) The Feynman diagram corresponding to the  $n$ -th order term ( $n \geq 1$ ) in the perturbative expansion of the truncated–connected quark propagator in an external gluon field  $A_a^\mu$ , in the eikonal approximation. [See Eq. (2.14) for the scalar case and Eqs. (3.12) and (3.16) for the fermion case.] b) The Feynman diagram which defines the  $n$ -th order perturbative term ( $n \geq 1$ ) of the gluon matrix element (4.1). Crosses represent insertions of the external gluon field  $A_a^\mu$ . The four–momenta  $q_1, \dots, q_n$  are taken to be flowing into the diagram.

Figure 1

a)



b)

